S20 PHYSICS 7B: Wang MT 2 Solutions

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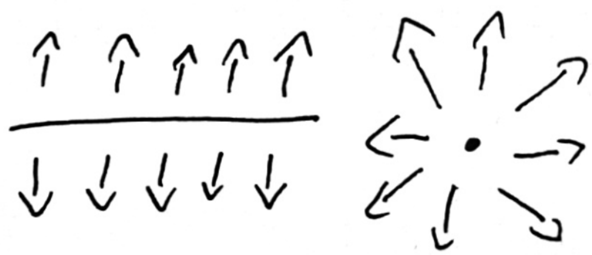
1 Problem 1

(a)



The left side shows a side-on view, while the right side shows a head-on view. The field is radial.

(b)

Clearly the flux through the two faces that the wire passes through is 0. By symmetry, the flux through each of the remaining four faces is identical. That is, if the total flux through the cube is Φ, then each of the four faces that are not intersected by the wire have flux $\Phi_f = \Phi/4$. To compute Φ, we use Gauss's Law:

$$\Phi = \int \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}. \tag{1.1}$$

We can explicitly compute the enclosed charge:

$$Q_{\text{enc}} = \lambda L, \tag{1.2}$$

so

$$\Phi_f = \frac{1}{4}\Phi = \frac{Q_{\text{enc}}}{4\epsilon_0} = \frac{\lambda L}{4\epsilon_0}. \tag{1.3}$$

Alternatively, we could directly compute the flux through a longer, but straightforward, calculation:

$$\Phi_f = \int \mathbf{E} \cdot d\mathbf{a} \tag{1.4}$$

$$= \int \frac{\lambda}{2\pi\epsilon_0 r} \cos\theta \, dx \, dy \tag{1.5}$$

$$= \frac{\lambda}{2\pi\epsilon_0} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \frac{1}{\sqrt{(L/2)^2 + y^2}} \frac{L/2}{\sqrt{(L/2)^2 + y^2}} \, dx \, dy \tag{1.6}$$

$$= \frac{\lambda L}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \frac{1}{(L/2)^2 + y^2} \, dx \, dy \tag{1.7}$$

$$= \frac{\lambda L}{4\pi\epsilon_0}(L)\left(\frac{\pi}{L}\right) \tag{1.8}$$

$$= \frac{\lambda L}{4\epsilon_0} \tag{1.9}$$

2 Problem 2

(a)

We begin by finding the field everywhere, which can be done in the standard way using Gauss's Law for a spherically symmetric system, so that the field is only radial:

$$\mathbf{E} = \begin{cases} 0, & r > c \\ \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}, & c \geq r \geq b \\ 0, b > r \end{cases} . \tag{2.1}$$

$$V(a) = -\int \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^a E\, dr \tag{2.2}$$

so we can compute:

$$V(r) = \begin{cases} 0, & r > c \\ \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{c}\right), & c \geq r \geq b \\ \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{c}\right), & b > r \end{cases} \tag{2.3}$$

In particular, note that the boundary conditions need to match at $r = b, c$.

(b)

By connecting the inner and outer shells, charge will flow until they have the same potential. Suppose the final charge on the outer shell is Q_c and the inner shell is Q_a, with the restriction that $Q_a + Q_c = -Q$, by charge conservation. The middle shell still has charge $Q_b = Q$. Now computing the potential everywhere:

$$V(r) = \begin{cases} 0, & r > c \\ \frac{Q+Q_a}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{c}\right), & c \geq r \geq b \\ \frac{Q+Q_a}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{Q_a}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right), & b > r \geq a \\ \frac{Q+Q_a}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{Q_a}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right), & a > r \end{cases} \tag{2.4}$$

Now equating the potentials at $r = c$ and $r = a$:

$$0 = \frac{Q + Q_a}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{Q_a}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right), \tag{2.5}$$

and solving for Q_a, we find:

$$Q_a = Q\frac{\frac{1}{b} - \frac{1}{c}}{\frac{1}{c} - \frac{1}{a}} = Q\frac{a(c-b)}{b(a-c)}, \tag{2.6}$$

and correspondingly

$$Q_c = -Q - Q_a = -Q\left(1 + \frac{a(c-b)}{b(a-c)}\right). \tag{2.7}$$

3 Problem 3

(a)

The sphere with a cavity can be treated as the superposition of two spheres: one with radius R and charge density ρ, and one with radius $3R/4$ and charge density $-\rho$. The electric field at any point is then just the superposition of the fields from each sphere and the point charge. Between $(R, 0, 0)$ and $(L, 0, 0)$, the field from each sphere and the point charge just points in the x direction. The field from the larger sphere is:

$$\mathbf{E}_1 = \frac{Q_1}{4\pi\epsilon_0 x^2}\hat{x}, \tag{3.1}$$

where $Q_1 = \frac{4}{3}\pi R^3 \rho$. The field from the smaller sphere is

$$\mathbf{E}_2 = \frac{Q_2}{4\pi\epsilon_0 (x - R/4)^2}\hat{x}, \tag{3.2}$$

where $Q_2 = -\frac{4}{3}\pi(3R/4)^3\rho = -\frac{9}{16}\pi R^3 \rho$. Finally, the field from the point charge is

$$\mathbf{E}_3 = \frac{Q}{4\pi\epsilon_0 (x - L)^2}\hat{x}. \tag{3.3}$$

Then the field is the sum of all three:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{4\pi R^3 \rho}{3x^2} - \frac{9\pi R^3 \rho}{16(x - R/4)^2} + \frac{Q}{(x - L)^2}\right)\hat{x}. \tag{3.4}$$

(b)

By Newton's Third Law, the force on the sphere with the cavity is equal in magnitude and opposite to its force on the point charge. Its force on the point charge is computed as:

$$\mathbf{F}_q = Q\mathbf{E}_s = Q(\mathbf{E}_1 + \mathbf{E}_2) = \frac{QR^3\rho}{3\epsilon_0}\left(\frac{1}{L^2} - \frac{27}{64(L - R/4)^2}\right)\hat{x}. \tag{3.5}$$

Then the force on the sphere with the cavity is:

$$\mathbf{F}_s = -\mathbf{F}_q = -\frac{QR^3\rho}{3\epsilon_0}\left(\frac{1}{L^2} - \frac{27}{64(L - R/4)^2}\right)\hat{x}. \tag{3.6}$$

4 Problem 4

(a)

Let E_1, E_2, E_3 be the electric field strengths in regions I, II, III, respectively. We can compute the field everywhere using superposition of the field from a sheet of charge with uniform surface charge density σ. Because the thick metal plate is a conductor, we must have $E_2 = 0$. Moreover, it must have $-Q$ total charge on its left side at $z = x$, and it must have Q total charge on its right side at $z = x + L$ in order to force $E_2 = 0$. Then setting $z = 0$ at the left plate and $z = d$ at the right, and $\sigma = Q/A$, we have

$$\mathbf{E}_1 = \frac{\sigma}{\epsilon_0}\hat{z}, \ \mathbf{E}_2 = 0, \ \mathbf{E}_3 = \frac{\sigma}{\epsilon_0}\hat{z}. \tag{4.1}$$

We integrate the electric field to find the potential difference:

$$|\Delta V| = \int_0^d \mathbf{E} \cdot d\mathbf{l} = E_1 x + E_2 L + E_3(d - x - L), \tag{4.2}$$

giving:

$$|\Delta V| = E_1 x + E_3(d - x - L) = \frac{\sigma}{\epsilon_0} x + \frac{\sigma}{\epsilon_0}(d - x - L) = \frac{\sigma}{\epsilon_0}(d - L). \tag{4.3}$$

(b)

The energy stored in the system is given by:

$$U = \frac{1}{2} C V^2 = \frac{Q^2}{2C}. \tag{4.4}$$

The capacitance can be computed by noting that the capacitor plates effectively form two capacitors in series. The capacitance of two sheets of charge Q and $-Q$ separated by distance z and with areas A is:

$$C = \frac{Q}{V} = \frac{A\epsilon_0}{z}. \tag{4.5}$$

Then we essentially have two capacitors of capacitance $C_1 = \frac{A\epsilon_0}{x}$ and $C_2 = \frac{A\epsilon_0}{d-x-L}$. Combining these in series:

$$C_{\text{eq}} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} = \left(\frac{x}{A\epsilon_0} + \frac{d - x - L}{A\epsilon_0}\right)^{-1} = \frac{A\epsilon_0}{d - L}, \tag{4.6}$$

and we hence find:

$$U = \frac{Q^2}{2C} = \frac{Q^2(d - L)}{2A\epsilon_0}. \tag{4.7}$$

(c)

The work needed to pull the plate out a long distance away is given by:

$$W = U_f - U_i, \tag{4.8}$$

where U_f is the energy of the final configuration and U_i is the energy of the initial configuration. We compute the U_i in part (b). To compute U_f, we are simply computing the energy of a single capacitor with separation distance d, which we know is:

$$C_f = \frac{A\epsilon_0}{d}. \tag{4.9}$$

Hence:

$$W = \frac{Q^2 d}{2A\epsilon_0} - \frac{Q^2(d - L)}{2A\epsilon_0} = \frac{Q^2 L}{2A\epsilon_0}. \tag{4.10}$$

5 Problem 5

(a)

To simplify the notation, let's set all resistors to have resistance R and batteries to have voltage difference V_0 and put in numbers at the end. Consider the point at the top of the loop, to the left of the battery. Consider the loop of resistors in the clockwise direction from this point. By Kirchhoff's loop law, the voltage drop across the loop must be 0. So tracing the voltage drops from our origin point:

$$V_0 + IR + IR + V_0 + IR + IR + V_0 + IR = 0, \tag{5.1}$$

where we have used the fact that the circuit, which is completely in series, has the same current through each resistor. Solving for the current above, we find:

$$I = -\frac{3V_0}{5R}. \tag{5.2}$$

We can now compute the voltage at any point in the circuit. From point A to point B, we trace the voltage drop as:

$$V_A + IR + V_0 + IR = V_B, \tag{5.3}$$

so

$$V_B - V_A = 2IR + V_0 = -\frac{6}{5}V_0 + V_0 = -\frac{1}{5}V_0 = -0.3\text{V}. \tag{5.4}$$

(b)

By introducing a short through the A and B terminals, we now have two new loops in the system: the loop L_1 on the right created by connecting A and B, and a large loop L_2 ignoring the 2 resistors and 1 battery between A and B. Let us first consider L_1 in the clockwise direction: applying the loop rules gives:

$$I_1R - V_0 + I_1R = 0 \quad \Longrightarrow \quad I_1 = \frac{V_0}{2R}. \tag{5.5}$$

Note that this is *not* the current running through A and B; this is only the current running through the two resistors and battery in this part of the loop. Now consider L_2 in the clockwise direction, for which applying the loop rules gives

$$I_2R + V_0 + I_2R + V_0 + I_2R = 0 \quad \Longrightarrow \quad I_2 = -\frac{2V_0}{3R}. \tag{5.6}$$

With these currents in hand, we can now applying the junction rule to (say) the upper junction:

$$I_{AB} = I_1 + I_2 = \frac{V_0}{2R} - \frac{2V_0}{3R} = -\frac{V_0}{6R} = -2.5\text{mA}. \tag{5.7}$$

By our choice of directions at the junction, the positive current direction is from A to B, so the negative sign implies the current goes from B to A.