# FALL 2020 FINAL EXAM SOLUTIONS

NIKHIL SRIVASTAVA + MATH 54 STUDENTS

The untimed section long answers are taken from the exam of Sophia Xiao, and appear starting on page 5.

## 1. True False Questions

(1) If $v_1, v_2, v_3, v_4 \in \mathbb{R}^5$ are linearly independent and $A$ is a $5\times 5$ matrix with $rank(A) = 3$ then $Av_1, Av_2, Av_3, Av_4 \in \mathbb{R}^5$ must be linearly dependent.

True. Since $Av_1, \ldots, Av_4 \in col(A)$ are four vectors in a subspace of dimension 3, they cannot be linearly independent.

(2) If $A$ is an $m \times n$ matrix with reduced row echelon form $R$, then the number of pivots in $R$ is equal to the number of nonzero singular values of $A$, counted with multiplicity.

True. Both are equal to the rank.

(3) If a $3 \times 2$ matrix $A$ has two nonzero singular values, then there is a unique least squares solution to $Ax = b$ for every $b \in \mathbb{R}^3$.

True. The condition on singular values means $A$ has rank 2, so its columns are linearly independent, which implies that $Ax = 0$ has a unique solution, implying uniqueness of the least squares solution.

(4) If a $3 \times 2$ matrix has orthonormal columns, then it must have orthonormal rows.

False. Consider $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(5) If $W$ and $H$ are 3 dimensional subspaces of $\mathbb{R}^5$ and $P$ and $Q$ are the standard matrices of the orthogonal projections onto them (respectively), then $PQ$ is the standard matrix of the orthogonal projection onto the subspace $W \cap H$.

False. $PQ$ may not be a projection matrix in general, in fact it may not even be symmetric! One specific example is $W = \text{span}\{e_1, e_2, e_3\}$ and $H = \text{span}\{e_1 + e_2, e_3 + e_4, e_5\}$.

(6) If $A$ is a $5 \times 5$ matrix with exactly four nonzero entries, then $rank(A) \leq 4$.

True. Row reduction never increases the number of zeros in a matrix, so the RREF of $A$ must have at most four pivots. An alternate proof is that $A$ has at most 4 nonzero columns, so its column space has rank at most 4.

(7) If $A$ is similar to $B$ and $B$ is symmetric then $A$ must be symmetric.

False. If $B = B^T$ and $A = PBP^{-1}$ for $P$ which is *not* orthogonal, then $A$ is not symmetric, since symmetric matrices are precisely those that are orthogonally diagonalizable.

(8) If $\sigma$ is a singular value of a square matrix $A$ then $\sigma^2$ must be a singular value of $A^2$.

False. 1 is a singular value of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ but $A^2 = 0$ has all singular values equal to zero.

(9) If $A$ is a $4 \times 4$ matrix then it can be written as $A = US$ for some orthogonal $U$ and symmetric $S$.

True. Let $A = U\Sigma V^T$ be the full SVD of $A$. Then $A = UV^TV\Sigma V^T = (UV^T)(V\Sigma V^T)$ since $V^TV = I$. The first matrix is orthogonal since a $V^T$ is orthogonal and a product of orthogonal matrices is orthogonal, and the second matrix is symmetric. This was the hardest question on the untimed portion of the exam.

(10) Let $F$ be the vector space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. If three functions $f(t), g(t), h(t) \in F$ are linearly dependent, then the vectors

$$\begin{bmatrix} f(0) \\ f'(0) \\ f''(0) \end{bmatrix}, \begin{bmatrix} g(0) \\ g'(0) \\ g''(0) \end{bmatrix}, \begin{bmatrix} h(0) \\ h'(0) \\ h''(0) \end{bmatrix} \in \mathbb{R}^3$$

must be linearly dependent.

True. $c_1 f + c_2 g + c_3 h = 0$ in $F$ implies $c_1 f' + c_2 g' + c_3 h' = 0$ as well as $c_1 f'' + c_2 g'' + c_3 h'' = 0$. Plugging in 0 yields the conclusion.

(11) Let $F$ be the vector space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. If three functions $f(t), g(t), h(t) \in F$ are linearly independent, then the vectors

$$\begin{bmatrix} f(0) \\ f'(0) \\ f''(0) \end{bmatrix}, \begin{bmatrix} g(0) \\ g'(0) \\ g''(0) \end{bmatrix}, \begin{bmatrix} h(0) \\ h'(0) \\ h''(0) \end{bmatrix} \in \mathbb{R}^3$$

must be linearly independent.

False. Consider the three functions $t^3, t^4, t^5$.

## 2. AM and PM Questions

Give an example of each of the following, explaining why it has the required property, or explain why no such example exists.

(1) Two $3 \times 3$ matrices $A, B$ such that $rank(A) = rank(B) = 1$ and

$$rank(A + B) = 3.$$

Does not exist. If $col(A) = \text{span}\{v_1\}$ and $col(A) = \text{span}\{v_2\}$ then every vector in $col(A + B)$ can be written as $c_1v_1 + c_2v_2$, so the latter has dimension at most 2.

(2) Two $2 \times 3$ matrices $A_1, A_2$ with nonnegative entries (i.e., $\geq 0$) such that the linear systems

$$A_1x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A_2x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

are consistent but the linear system

$$(A_1 + A_2)x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is inconsistent.

There are many examples. The key point is that the $x_1$ which solves $A_1x_1 = b$ need not have anything to do with the one that solves $A_2x_2 = b$ — many people missed this and mistakenly argued that you could use the same solution. For a concrete example, consider

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, (A_1 + A_2) = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

(3) A $3 \times 3$ symmetric matrix $A$ with all nonnegative entries (i.e., $a_{ij} \geq 0$) which has at least one *strictly negative* eigenvalue (i.e., $\lambda < 0$).

Many examples. Easiest is to start with a $2 \times 2$ matrix with the required property, such as $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ which has a negative determinant, and embed it in the corner of a $3 \times 3$ matrix: $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which just adds an extra zero eigenvalue.

(4) Two $3 \times 3$ symmetric matrices $A, B$ such that the product $AB$ is not diagonalizable.

Again, easiest to first look for a $2 \times 2$ example. One instance is $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (which 'swaps' the entries) and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ with $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, our favorite non-diagonalizable matrix. Again, just add zeros to get a $3 \times 3$ example $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(5) Two $3 \times 3$ matrices $A, B$ with nonnegative entries (i.e., $\geq 0$) such that $A$ and $B$ are diagonalizable but $A + B$ is not diagonalizable.

Many examples. The first thing to remember is that diagonalizable matrices must have multiple eigenvalues, so $A + B$ must have this property. It is easy to know the eigenvalues of upper triangular matrices, so let's work with those. One then has the the example

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(6) Two $3 \times 3$ matrices $A, B$ such that $rank(A) = rank(B) = 2$ and $AB = 0$.

Does not exist. Suppose $AB = 0$ and $B$ has rank 2 then $B$ has two linearly independent columns, say $b_1, b_2$, and $AB = 0$ implies $Ab_1 = Ab_2 = 0$. But now there are two linearly independent vectors in $null(A)$, which implies $rank(A) \leq 1$, a contradiction.

(7) Two $2 \times 2$ matrices $A_1, A_2$ with all nonnegative entries (i.e., $\geq 0$) such that the linear systems

$$A_1 x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$A_2 x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

are consistent, but the system

$$A_1 A_2 x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is inconsistent.

Many examples. Take $A_1 = A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then $col(A_2)$ does not contain $e_1$, so $A_1 A_2 x = b$ is consistent for the $b$ above.

(8) A $3 \times 3$ matrix $A$ with characteristic polynomial equal to

$$\det(A - \lambda I) = -\lambda^2(\lambda - 1)$$

and a singular value equal to

$$2$$

.

Many examples. The characteristic polynomial tells us that the eigenvalues are $0, 0, 1$. Again, easiest to work with upper triangular matrices. To get a singular value of 2 we should have $A^T A$ having an eigenvalue of 4. It would be easiest to determine this if $A^T A$ were diagonal, i.e., if $A$ had orthogonal columns. This leads us to the example

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(9) A $3 \times 3$ matrix $A$ with characteristic polynomial equal to

$$\det(A - \lambda I) = -\lambda^2(\lambda - 2)$$

and a singular value equal to 1.

Many examples. Same reasoning as above yields

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

# Q3.1

let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

Define $\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + 2u_2 v_2 + u_3 v_3$.

This is an inner product because it satisfies the following axioms :

① $\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + 2u_2 v_2 + u_3 v_3 = \langle \vec{v}, \vec{u} \rangle$

② $\langle \vec{u} + \vec{v}, \vec{w} \rangle = (u_1 + v_1) w_1 + 2(u_2 + v_2) w_2 + (u_3 + v_3) w_3$

$= u_1 w_1 + 2u_2 w_2 + u_3 w_3 + v_1 w_1 + 2v_2 w_2 + v_3 w_3$

$= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ (where $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$)

③ $\langle c\vec{u}, \vec{v} \rangle = cu_1 v_1 + 2cu_2 v_2 + cu_3 v_3 = c(u_1 v_1 + 2u_2 v_2 + u_3 v_3)$

$= c\langle \vec{u}, \vec{v} \rangle$

④ $\langle \vec{u}, \vec{u} \rangle = u_1^2 + 2u_2^2 + u_3^2 \geq 0$ and

$u_1^2 + 2u_2^2 + u_3^2 = 0 \longleftrightarrow u_1 = u_2 = u_3 = 0$ .

Furthermore,

$\langle \vec{v}_1, \vec{v}_2 \rangle = 1(0) + 2(1)(1) + (-1)(2) = 0$

$\therefore$ $\boxed{\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + 2u_2 v_2 + u_3 v_3}$ is such an inner product .

# Q 3.2

$\boxed{\text{No such operator exists.}}$

$$T(y) = ay' + by$$

$$T^2(y) = a(ay' + by)' + b(ay' + by)$$

$$= a(ay'' + by') + b(ay' + by)$$

$\ker(T)$ : $ay' + by = 0$

$\ker(T)$ is all the solutions to the above 1st-order linear ODE. Therefore $\ker(T)$ has 1 dimension.

$\ker(T^2)$: soln. to $a^2y'' + aby' + aby' + b^2y$

$\ker(T^2)$ has 2 dimensions.

Since $\dim(\ker(T)) = 1$ and $\dim(\ker(T^2)) = 2$,

$\ker(T) \neq \ker(T^2)$ for all $T$ of the form

$T = a\frac{d}{dt} + bI$.

# Q4

a). Wronskian

$$W[e^t, \sin(2t), \cos(3t)](t)$$

$$= \begin{vmatrix} e^t & \sin(2t) & \cos(3t) \\ e^t & 2\cos(2t) & -3\sin(3t) \\ e^t & -4\sin(2t) & -9\cos(3t) \end{vmatrix}$$

$$= e^t\left(-18\cos(2t)\cos(3t) - 12\sin(3t)\sin(2t)\right)$$
$$- \sin(2t)\left(-9e^t\cos(3t) + 3e^t\sin(3t)\right)$$
$$+ \cos(3t)\left(-4e^t\sin(2t) + 2e^t\cos(2t)\right)$$

$$= e^t\Big[-18\cos(2t)\cos(3t) - 12\sin(3t)\sin(2t) + 9\sin(2t)\cos(3t)$$
$$- 3\sin(2t)\sin(3t) - 4\cos(3t)\sin(2t) + 2\cos(3t)\cos(2t)\Big]$$

$$= e^t\left[-16\cos(2t)\cos(3t) - 15\sin(2t)\sin(3t) + 5\sin(2t)\cos(3t)\right]$$

$$W[e^t, \sin(2t), \cos(3t)](0) = 1[-16] = -16 \neq 0$$

Since the Wronskian is nonzero at $t=0$,
$\{e^t, \sin(2t), \cos(2t)\}$ are linearly independent.

b). Since $\{e^t, \sin(2t), \cos(2t)\}$ are linearly independent, they form a basis for $W$. Let's call this basis $B$.

Let $\vec{b}_1 = e^t$, $\vec{b}_2 = \sin(2t)$, $\vec{b}_3 = \cos(2t)$

$$[T]_B = \left[\, [T(\vec{b}_1)]_B \;\; [T(\vec{b}_2)]_B \;\; [T(\vec{b}_3)]_B \,\right]$$

$$T(\vec{b}_1) = T(e^t) = e^t - ae^t = (1-a)e^t = (1-a)\vec{b}_1$$

$$T(\vec{b}_2) = T(\sin(2t)) = -4\sin(2t) - a\sin(2t) = (-4-a)\vec{b}_2$$

$$T(\vec{b}_3) = T(\cos(3t)) = -9\cos(3t) - a\cos(3t) = (-9-a)\vec{b}_3$$

$$[T]_B = \begin{bmatrix} 1-a & 0 & 0 \\ 0 & -4-a & 0 \\ 0 & 0 & -9-a \end{bmatrix}$$ is invertible when there are 3 pivots.

$$1-a \neq 0, \quad -4-a \neq 0, \quad -9-a \neq 0$$

$\boxed{a \neq 1, -4, -9}$ all $a$ except those

# Q5

a). let $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ be a basis for $M_2$.

If $X = X^T$, then $X - X^T = 0$. Therefore, to find all $X$ s.t. $X = X^T$ we can just find $\ker(T)$ where $T(X) = X - X^T$.

$$T(\vec{b_1}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$T(\vec{b_2}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \vec{b_2} - \vec{b_3}$$

$$T(\vec{b_3}) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \vec{b_3} - \vec{b_2}$$

$$T(\vec{b_4}) = 0$$

$$[T]_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\ker([T]_B) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\ker(T) = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

dimension 3

$\therefore \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis of $W$.

b). $AXA^T \in M_2$ because $A$ is $2 \times 2$. Now we just have to show ⅌ that $(AXA^T)^T = AXA^T$. if $X = X^T$.

pf: $(AXA^T)^T = A^{TT} X^T A^T = AX^T A^T = AXA^T$ ✓

$\therefore AXA^T \in W$

# Q5 (cont.)

c). Let C be the basis of W found in part a).

$$T(\vec{c_1}) = T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \vec{c_1} + \vec{c_2} - \vec{c_3}$$

$$T(\vec{c_2}) = T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \vec{c_1} + \vec{c_2} + \vec{c_3}$$

$$T(\vec{c_3}) = T\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = 2\vec{c_1} - 2\vec{c_2}$$

$$\cancel{[W} \quad [T]_C = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ -1 & 1 & 0 \end{bmatrix}$$

Find eigenvalues of $[T]_C$:

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & 1-\lambda & -2 \\ -1 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda+\lambda^2+2) - (-\lambda-2) + 2(1+1-\lambda)$$

$$= -\lambda^3 + 2\lambda^2 - 3\lambda + 2 - \lambda + 6$$

$$= -\lambda^3 + 2\lambda^2 - 4\lambda + 8 = 0$$

It is easy to see that $\lambda = 2$.

Find eigenspace of $\lambda = 2$:

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & -2 \\ -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$E_{\lambda=2}$ : $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Eigenvector: $\vec{c_1} + \vec{c_2} = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$ has eigenvalue 2.

CHECK: $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ ✓

# Q6

a). $A^TA = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Finding singular values:

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0$$

$$= \lambda^2 - 4\lambda + 3 = 0$$

$$= (\lambda-1)(\lambda-3) = 0$$

$\sigma_1 = \sqrt{3}$

$\sigma_2 = 1$

$\Rightarrow \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}$

Finding V:

$\vec{v_1}: \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\vec{v_2}: \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad \vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$$V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Finding U:

$$\vec{u_1} = \frac{A\vec{v_1}}{\sqrt{3}} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

$$\vec{u_2} = \frac{A\vec{v_2}}{1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$U = \begin{bmatrix} 2/\sqrt{6} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} \\ -1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}$$

$$\boxed{A = U\Sigma V^T = \begin{bmatrix} 2/\sqrt{6} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} \\ -1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}$$

## Q6 (cont)

b). $\text{proj}_{Col(A)} \vec{x} = UU^T\vec{x}$

because $U$ is an orthonormal basis for Col(A).

$$P = UU^T = \begin{bmatrix} 2/\sqrt{6} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} \\ -1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4/6 & -2/6 & -2/6 \\ -2/6 & 1/6 + 1/2 & 1/6 - 1/2 \\ -2/6 & 1/6 - 1/2 & 1/6 + 1/2 \end{bmatrix} = \boxed{\begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}}$$

c). ~~lex~~ Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then $A\vec{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ -x_2 \\ -x_1 \end{bmatrix}$

$$|A\vec{x}| = \sqrt{x_1^2 + x_2^2 + 2x_1x_2 + x_2^2 + x_1^2}$$
$$= \sqrt{2x_1^2 + 2x_2^2 + 2x_1x_2}$$
$$|\vec{x}| = \sqrt{x_1^2 + x_2^2}$$

✩ Alternatively, note that $|A\vec{v}_1| = \sigma_1$

so $|A\vec{v}_2| = 1$

and $|\vec{v}_2| = 1$

$\therefore \vec{x} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$$|A\vec{x}| \le |\vec{x}| \longleftrightarrow |A\vec{x}|^2 \le |\vec{x}|^2 \longleftrightarrow 2x_1^2 + 2x_2^2 + 2x_1x_2 \le x_1^2 + x_2^2$$

$$\longleftrightarrow x_1^2 + x_2^2 + 2x_1x_2 \le 0 \longleftrightarrow (x_1 + x_2)^2 \le 0.$$

This only occurs when $x_1 = -x_2$. Let $x_1 = 1$.

$\boxed{\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$ $\quad |A\vec{x}| \le |\vec{x}|$ for all $\begin{bmatrix} a \\ -a \end{bmatrix}$

CHECK: $A\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\quad |A\vec{x}| = \sqrt{2}$, $|\vec{x}| = \sqrt{2}$ $\quad |A\vec{x}| \le |\vec{x}|$ ✓

Q7

a). $y'' - 2y' + 2y = 0$

$r^2 - 2r + 2 = 0 \implies r = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

Plug into formula for complex roots:

$y \in \text{span}\{e^t \cos t,\ e^t \sin t\}$

General Soln: $\boxed{y = c_1 e^t \cos t + c_2 e^t \sin t}$

b). We already found $y^{(h)} = c_1 e^t \cos t + c_2 e^t \sin t$

To find the particular soln, we guess $y^{(p)} = At + B$.

$y' = A$

$y'' = 0 \implies 0 - 2A + 2(At + B) = t + 1$

$-2A + 2At + 2B = t + 1$

$\Downarrow$

$2A = 1 \qquad -2A + 2B = 1$

$A = 1/2 \qquad -1 + 2B = 1 \qquad B = 1$

$y^{(p)} = \frac{1}{2}t + 1$

$\boxed{y = c_1 e^t \cos t + c_2 e^t \sin t + \frac{1}{2}t + 1}$

c). $y'(t) = c_1(-e^t \sin t + e^t \cos t) + c_2(e^t \cos t + e^t \sin t) + \frac{1}{2}$

$y'(0) = \frac{1}{2} + c_1 + c_2 = 1 \rightarrow c_1 + c_2 = 1/2 \rightarrow c_2 = \frac{1}{2} + 1$

$y(0) = c_1 + 1 = 0 \rightarrow c_1 = -1$

$\boxed{y = -e^t \cos t + \frac{3}{2} e^t \sin t + \frac{1}{2}t + 1}$

# Q8

a). Find eigenvalues of $A$:

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\left[(2-\lambda)^2-1\right] - (2-\lambda-1) + (1-2+\lambda)$$

$$= (2-\lambda)(\lambda^2-4\lambda+3 + \lambda - 1 + \lambda - 1$$

$$= -\lambda^3 + 6\lambda^2 - 9\lambda + 4$$

$$= -(\lambda-1)^2(\lambda-4) = 0$$

$\hookrightarrow \quad \lambda = 1, 4$

$\lambda = 1$ : $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda = 4$ : $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Plug into formula for $\vec{x}$:

$$\boxed{\vec{x}(t) = c_1 e^{t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

b). $\vec{x}'(t) = (A^3 - A^2 + I)\,\vec{x}(t)$

let $\vec{v}$ be an eigenvector of $A$ & $\lambda$ be the corresponding eigenvalue.

$(A^3 - A^2 + I)\vec{v} = A^3\vec{v} - A^2\vec{v} + I\vec{v}$

$= \lambda^3\vec{v} - \lambda^2 v + \vec{v} = (\lambda^3 - \lambda^2 + 1)\,\vec{v}$

$\therefore$ $\vec{v}$ is also an eigenvector of $(A^3 - A^2 + I)$ and $(\lambda^3 - \lambda^2 + 1)$ is the corresponding eigenvalue.

$\lambda = 1 \rightsquigarrow \lambda' = 1 - 1 + 1 = 1$

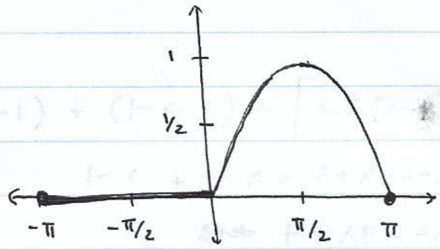
$\lambda = 4 \rightsquigarrow \lambda' = 64 - 16 + 1 = 49$

Since $(A^3 - A^2 + I)$ has at most 3 LI eigenvectors, the eigenvectors of $A$ are all of them.

$$\boxed{\vec{x}(t) = c_1 e^{t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{49t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

# Q9

a).




b). $$\frac{a_0}{2} = \frac{\langle 1, f\rangle}{2\pi} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi} \sin x\,dx = \frac{1}{2\pi}\Big[-\cos x\Big]_0^{\pi} = \frac{1}{2\pi}(1 - -1)$$

$$= \frac{1}{\pi}$$

$$a_n = \frac{\langle \cos(nx), f\rangle}{|\cos(nx)|^2} = \frac{1}{\pi}\int_0^{\pi} \sin x \cos(nx)\,dx = \frac{1}{2\pi}\int_0^{\pi} \sin(nx+x) + \sin(x-nx)\,dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(nx+x)}{n+1} - \frac{\cos(x-nx)}{1-n}\right]_0^{\pi}$$ ★ when $n \neq 1$

$$= \frac{1}{2\pi}\left[\frac{-\cos(\pi n+\pi)}{n+1} - \frac{\cos(\pi - n\pi)}{1-n} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\cos(\pi n)}{n+1} + \frac{\cos(\pi n)}{1-n} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^n}{n+1} + \frac{(-1)^n}{1-n} + \frac{1}{n+1} - \frac{1}{n-1}\right] = \frac{1}{2\pi}\left[\frac{(-1)^n - n(-1)^n + (-1)^n + (-1)^n n + 1 - n + n + 1}{1-n^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2(-1)^n + 2}{1-n^2}\right] = \frac{1}{\pi}\left(\frac{(-1)^n + 1}{1-n^2}\right)$$ ★ when $n \neq 1$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x\,dx = \frac{1}{\pi}\left[\frac{\sin^2 x}{2}\right]_0^{\pi} = 0$$ ★ when $n=1$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\sin(nx)\,dx = \frac{1}{2\pi}\int_0^{\pi} \cos(x-nx) - \cos(x+nx)\,dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(x-nx)}{1-n} - \frac{\sin(x+nx)}{1+n}\right]_0^{\pi}$$ ★ when $n \neq 1$

$$= \frac{1}{2\pi}\left[\frac{\sin(\pi - n\pi)}{1-n} - \frac{\sin(\pi + n\pi)}{1+n}\right] = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2(x)\,dx = \frac{1}{\pi}\left[\frac{-\sin(2x) - 2x}{4}\right]_0^{\pi} = \frac{1}{2}$$

$(\longrightarrow)$

Q9 (cont).

$$\boxed{a_n = \begin{cases} 0 & n=1 \\ \dfrac{(-1)^n+1}{\pi(1-n^2)} & n \neq 1 \end{cases} \qquad b_n = \begin{cases} 1/2 & n=1 \\ 0 & n \neq 1 \end{cases}}$$

$$\frac{a_0}{2} = \frac{1}{\pi}$$

c). By the best approximation thm., $\text{proj}_{F_{odd}} f$ is the closest vector to $f$ in $F_{odd}$.

In other words, $\hat{f} = \text{proj}_{F_{odd}} f$ minimizes $\|f-\hat{f}\|^2$.

Note that $\{\sin(nx)\}_{n=1}^{\infty}$ is an orthogonal basis of $F_{odd}$.

Therefore,

$$\hat{f} = \text{proj}_{F_{odd}} f = \frac{\langle \sin x, f \rangle}{|\sin x|^2} \sin x + \frac{\langle \sin(2x), f \rangle}{|\sin(2x)|^2} \sin(2x) + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{\langle \sin(nx), f \rangle}{|\sin(nx)|^2} \sin(nx).$$

We already found the coefficient of $\sin(nx)$ in part b). as $b_n$.
Since $b_1 = 1/2$ and $b_n$ is 0 everywhere else.

$$\boxed{\hat{f} = \frac{1}{2}\sin(x).}$$