Due	For	Available from	Until
-	1 student	Apr 10 at 11:10am	Apr 10 at 1:25pm
-	1 student	Apr 9 at 10pm	Apr 9 at 10:35pm
-	1 student	Apr 10 at 11:10am	Apr 10 at 2:10pm
-	1 student	Apr 10 at 11:10am	Apr 10 at 12:40pm

Preview

Correct answers are hidden.

Score for this quiz: **25** out of 25
Submitted Apr 15 at 10:42pm
This attempt took 3 minutes.

Question 1

1 / 1 pts

A dynamical system model for an epidemic with total population $N = S + I + R$, where S is the number of susceptible individuals, I is the number of infected, and R is the number of recovered, is modeled by

$$\frac{d}{dt}S = -\beta\frac{IS}{N}$$
$$\frac{d}{dt}I = \beta\frac{IS}{N} - \gamma I$$
$$\frac{d}{dt}R = \gamma I$$

Here, we use real numbers since integer granularity is not required. Consider the situation before the onset of the epidemic, with $S = N$, $I = 0$, and $R = 0$. The linearized state-space model is given by

$$\frac{d}{dt}\begin{bmatrix}\tilde{s}\\ \tilde{i}\\ \tilde{r}\end{bmatrix} = A\begin{bmatrix}\tilde{s}\\ \tilde{i}\\ \tilde{r}\end{bmatrix},$$

where the lower case variables with tildes are the linearized variables for the model. Then, the matrix A is given by:

- ○ $A = \begin{bmatrix}-\beta & -\beta & 0\\ \beta & \beta-\gamma & 0\\ 0 & \gamma & 0\end{bmatrix}$
- ○ $A = \begin{bmatrix}0 & -\beta & 0\\ 0 & \gamma-\beta & 0\\ 0 & \gamma & 0\end{bmatrix}$
- ◉ $A = \begin{bmatrix}0 & -\beta & 0\\ 0 & \beta-\gamma & 0\\ 0 & \gamma & 0\end{bmatrix}$
- ○ $A = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$
- ○ $A = \begin{bmatrix}-\beta & -\beta & 0\\ 0 & \beta-\gamma & 0\\ 0 & \gamma & 0\end{bmatrix}$

Question 2

1 / 1 pts

A system $\frac{d}{dt}\vec{x} = A\vec{x} + B\vec{u}$ has controllability matrix $\mathcal{C} = [B \quad AB \quad \dots \quad A^{n-1}B]$.

Suppose that $\vec{z} = T\vec{x}$, where T is an invertible matrix. What is the controllability matrix for the system resulting from this change of coordinates?

- (●) $T\mathcal{C}$
- ( ) $T\mathcal{C}T^{-1}$
- ( ) $\mathcal{C}T^{-1}$
- ( ) $\mathcal{C}$
- ( ) $T^{-1}\mathcal{C}$

Question 3

1 / 1 pts

Given the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

Which of the following are true statements about the Singular Value Decomposition (SVD) of A?

1. All eigenvalues λ_i of $AA^\top$ are identical to each other.
2. Non zero singular values are $\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1$.
3. Removing the last row of A doesn't change the non-zero singular values.

- ( ) 1 and 2 only.
- (●) 2 and 3 only.
- ( ) 1 and 3 only.

- 1 only.
- 1, 2, and 3.

Question 4

1 / 1 pts

Which of the following statements about the Singular Value Decomposition (SVD) is true when written in the form $A = \sigma_1 \vec{u_1}\vec{v_1}^\top + \sigma_2 \vec{u_2}\vec{v_2}^\top + \dots$? Assume that all σ_i, the singular values, are non-zero.

- (selected) $\{\vec{u_1}, \vec{u_2}, \dots\}$ is an orthonormal basis for the column space of A.
- The singular values, σ_i, are real numbers of arbitrary sign.
- The SVD separates a rank r matrix A into a sum of $r-1$ rank 1 matrices.
- The SVD of a matrix A is unique.
- None of the others.

Question 5

1 / 1 pts

The dynamics of an epidemic, with a fixed population N are sometimes modeled with a state-space model of the form:

$$\frac{d}{dt}S = -\beta\frac{IS}{N}$$
$$\frac{d}{dt}I = \beta\frac{IS}{N} - \gamma I$$
$$\frac{d}{dt}R = \gamma I$$

where S is the number of susceptible individuals, I is the number of infected individuals , R is the number of recovered individuals, and $N = S + I + R$ is the total population. Although numbers of individuals are integer valued, we use real numbers in this exercise since integer granularity is not needed. Positive constants β and γ parametrize the epidemic dynamics.

How many equilibrium points does the epidemic dynamics of the model above have?

- ○ 3
- ◉ Infinitely many
- ○ 1
- ○ 2
- ○ 0

Any point with $I = 0$ is an equilibrium point. There are infinitely many such choices.

Question 6

1 / 1 pts

When the system $\frac{d}{dt}\vec{x} = A\vec{x}$ is discretized at a certain sampling period, the resulting discrete-time state space model is $\vec{x}_d(t+1) = A_d\vec{x}_d(t)$.

What is the state space model when $\frac{d}{dt}\vec{x} = 2A\vec{x}$ is discretized at the same sampling period?

- $\vec{x}_d(t+1) = 2A_d\vec{x}_d(t)$
- $\vec{x}_d(t+1) = A_d^2\vec{x}_d(t)$
- $\vec{x}_d(t+1) = (A_d + 2I)\vec{x}_d(t)$
- $\vec{x}_d(t+1) = (A_d^2/2 + I)\vec{x}_d(t)$
- Not enough information to determine

Question 7

1 / 1 pts

Suppose the following linear dynamical system is controllable:

$$\frac{d}{dt}\vec{x} = \mathbf{A}\vec{x} + \vec{b}_1 u$$

Which additional conditions are necessary for the following system to be controllable?

$$\frac{d}{dt}\vec{x} = \mathbf{A}\vec{x} + \mathbf{B}\vec{u}$$

where $\mathbf{B} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}$.

- The system $\frac{d}{dt}\vec{x} = \mathbf{A}\vec{x} + \vec{b}_2 u$ must also be controllable.
- The system cannot be controllable under any conditions.

None, the system is already controllable.

The controllability matrix of the system can be written as

$\mathbf{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A^{n-1}B} \end{bmatrix}$, where n is the number of state variables.

We can rewrite the controllability matrix as

$\mathbf{C} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \mathbf{A}\vec{b}_1 & \mathbf{A}\vec{b}_2 & \dots & \mathbf{A^{n-1}}\vec{b}_1 & \mathbf{A^{n-1}}\vec{b}_2 \end{bmatrix}$.

Because the first system is already controllable, we know that the matrix $\begin{bmatrix} \vec{b}_1 & \mathbf{A}\vec{b}_1 & \dots & \mathbf{A}^{n-1}\vec{b}_1 \end{bmatrix}$ has rank n, so $\mathbf{C}$ must have rank n as well.

$\mathbf{A}$ and $\mathbf{B}$ have orthogonal columns.

$\vec{b}_1$ and $\vec{b}_2$ must be orthogonal.

Question 8

1 / 1 pts

Suppose we have a linear dynamical system $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$

where $\vec{x}(t) \in \mathbb{R}^n$ and $\vec{u}(t) \in \mathbb{R}^m$.

Which of the following are necessarily true:

I. $\vec{x} = 0$ is an equilibrium point for $\vec{u} = 0$.

II. For any given input $\vec{u}$, there must exist a unique equilibrium point $\vec{x}^*$.

III. Suppose $(\vec{x}^*, \vec{u}^*)$ is an equilibrium point, $\vec{x}(0) = \vec{x}^*$, and $\vec{u}(t) = \vec{u}^*$ for all $t \geq 0$. Then $\vec{x}(t)$ is constant for $t \geq 0$.

IV. If A is invertible, there exists an input for which there are no equilibrium points.

V. If $\vec{x}_1^*$ and $\vec{x}_2^*$ are equilibrium points for $\vec{u} = 0$, $\vec{x}_1^* + \vec{x}_2^*$ is also an equilibrium point.

- I only.
- II, III, IV
- (selected) I, III, V
- I, II, III, IV
- I, II, III, IV, V

I, III, and V are correct.

Note that:

I. If we plug in $\vec{x}(t) = 0$ and $\vec{u}(t) = 0$ to the right hand side of the system equation, we see that $\frac{d}{dt}\vec{x}(t) = 0$, showing that it is an equilibrium point.

II. Note that the equilibrium point isn't necessarily unique. We are seeking solutions for $A\vec{x} = -B\vec{u}$. If A is non-singular, there only exist solutions if $B\vec{u} \in \text{Range}(A)$. There can also be several solutions of the form $\vec{x}_p + \vec{x}_h$, where $\vec{x}_h \in \textbf{null}(A)$.

III. This is true. If we start at an equilibrium point, $\frac{d}{dt}\vec{x}(t) = 0$ for all $t \geq 0$, so $\vec{x}(t)$will be a constant.

IV. This is false. If A is invertible, $\vec{x} = -A^{-1}B\vec{u}$ will be an equilibrium point.

Question 9

1 / 1 pts

Consider the discrete time system

$$\vec{x}(k+1) = A\vec{x}(k) + \vec{b}u(k)$$

with $\vec{x}(\cdot) \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times3}$, and $\vec{b} \in \mathbb{R}^3$.

Suppose that the system is controllable from the origin $\vec{x}(0) = 0$ in 10 steps. That is, one can design a control sequence $\{u(0), u(1), \ldots, u(9)\}$ to reach any target state $\vec{x}^* = \vec{x}(10)$ in 10 steps. Which of the following is true?

- For any target state $\vec{x}^*$, one can find an initial condition $\vec{x}(0)$ and a two step input sequence $\{u(0), u(1)\}$ to reach $\vec{x}^*$.
- None of the other answers is correct.
- Any state $\vec{x}^*$ can be also be reached with a shorter input sequence $\{u(0), u(1)\}$ in two steps.
- The state $\vec{x}^*$ cannot be reached from the origin in 9 steps with any possible sequence $\{u(0), u(1), \ldots, u(8)\}$.
- The input sequence $\{u(0), u(1), \ldots, u(9)\}$ to reach $\vec{x}^*$ is unique.

Suppose we want

$$\vec{x}(2) = \vec{x}^* = A\vec{x}(1) + \vec{b}u(1)$$
$$= A(A\vec{x}(0) + \vec{b}u(0)) + \vec{b}u(1)$$
$$= A^2\vec{x}(0) + A\vec{b}u(0) + \vec{b}u(1)$$
$$= A^2\vec{b}\alpha + A\vec{b}u(0) + \vec{b}u(1)$$

In the last line, we choose $\vec{x}(0) = \alpha\vec{b}$ for some scalar α.

We can rewrite this as

$$\vec{x}^* = \begin{bmatrix} A^2\vec{b} & A\vec{b} & \vec{b} \end{bmatrix} \begin{bmatrix} \alpha \\ u(0) \\ u(1) \end{bmatrix}$$

Since the system is controllable, $\begin{bmatrix} A^2\vec{b} & A\vec{b} & \vec{b} \end{bmatrix}$ has rank 3, and since we can choose $\alpha, u(0), u(1)$, and state $\vec{x}^*$ can be reached.

Question 10

1 / 1 pts

How many non-zero singular values does the following matrix A have?

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \\ 4 & 4 & 8 \\ 5 & 1 & 2 \end{bmatrix}$$

- ○ 1
- ○ 5
- ◉ 2

This is a rank 2 matrix. Because the third column is linearly dependent on the second column. So, the number of non-zero singular values will be 2.

- 4
- 3

Question 11

1 / 1 pts

Suppose we have the relation $\vec{y} = D\vec{p} + \vec{e}$, as seen from lecture. In order to determine $\vec{\hat{p}}$, the least squares estimate, which of the following assumptions were made?

- D is diagonal.
- $D^\top D$ is invertible.
- $\vec{e}$ is orthogonal to $\vec{y}$.
- None of the others assumptions.
- $D^\top$ is invertible.

Question 12

1 / 1 pts

Consider the scalar system $x(t+1) = bu(t) + e(t)$, where, b is the only unknown parameter and $e(t)$ is a disturbance term. Suppose, we apply the input, $u(0) = u(1) = u(2) = u(3) = 1$and observe the resulting state trajectory to obtain a least-squares estimate $\hat{b}$ for b. Which of the following state trajectories would result in the estimate $\hat{b}$ = 1?

- $x(1) = 1.1, x(2) = 0.9, x(3) = 1.2, x(4) = 1$
- $x(1) = 0.1, x(2) = 0.9, x(3) = 1.7, x(4) = 1.2$
- $x(1) = 0.1, x(2) = 1.9, x(3) = 1, x(4) = 0.9$
- $x(1) = 1.2, x(2) = 0.9, x(3) = 0.6, x(4) = 1.0$
- $x(1) = 0.1, x(2) = 1.1, x(3) = 1.9, x(4) = 0.9$

According to the general answer comment,

$x(1) + x(2) + x(3) + x(4) = 4$, which is true in this case.

$$\begin{bmatrix} x(1) \\ x(2) \\ x(3) \\ x(4) \end{bmatrix} = \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \end{bmatrix} b + \begin{bmatrix} e(0) \\ e(1) \\ e(2) \\ e(3) \end{bmatrix}$$

So,

$\vec{y} = Db + \vec{e}$, where,

$$\vec{y} = \begin{bmatrix} x(1) \\ x(2) \\ x(3) \\ x(4) \end{bmatrix}, D = \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, e = \begin{bmatrix} e(0) \\ e(1) \\ e(2) \\ e(3) \end{bmatrix}$$

The least-squares estimate of b is,

$\hat{b} = (D^T D)^{-1} D^T \vec{y} =$

$$(\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix})^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x(1) \\ x(2) \\ x(3) \\ x(4) \end{bmatrix}$$

$$\hat{b} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x(1) \\ x(2) \\ x(3) \\ x(4) \end{bmatrix}$$

$= \frac{1}{4}(x(1) + x(2) + x(3) + x(4))$

Question 13

1 / 1 pts

Which of the following are true about the Singular Value Decomposition (SVD)?

1. If a square matrix Q is orthonormal ($QQ^\top = I$), then its singular values are all 1.
2. A matrix with rank r will have exactly r singular values greater than 0.
3. Every real matrix has an SVD.

- 1 only.
- 2 and 3 only.
- 1 and 2 only.
- (selected) 1, 2, and 3.
- 1 and 3 only.

Question 14

1 / 1 pts

Consider a linear system, $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$, where $\vec{x}(t) \in \mathbb{R}^n$ and $\vec{u}(t) \in \mathbb{R}^m$.

Which of the the following conditions can, on its own, determine whether the system is **controllable or not**?

I.	$m < n$
II.	$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
III.	$m = n$ and B is invertible
IV.	$AB = 0$ and $m < n$
V	$\textbf{rank}(A) = n$

- (selected) II, III, and IV only

- I, II, III, IV, and V
- I, III, and V only
- I, II, III, and IV only
- II and III only

Question 15

1 / 1 pts

Consider the discrete time dynamical system

$$y(k+1) = b_1 u(k) + b_2 u(k-1) + e(k),$$

where $e(k)$ accounts for additive noise, and we get to measure the $y(\cdot)$ and the $u(\cdot)$ data sequences exactly. We set up an estimation scheme to estimate the unknown real parameters b_1, and b_2:

$$\begin{bmatrix} u(1) & u(0) \\ u(2) & u(1) \\ \vdots & \vdots \\ u(N) & u(N-1) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} y(2) \\ y(3) \\ \vdots \\ y(N+1) \end{bmatrix}.$$

Suppose that $u(k) = \lambda^k$. For this input, what is the minimum number of steps, i.e. samples of $y(\cdot)$, needed to uniquely estimate the parameters b_1 and b_2?

- 2
- 1
- 4
- (selected) Cannot be uniquely estimated, no matter how many samples
- 3

With the provided input, the first column of $\begin{bmatrix} u(1) & u(0) \\ u(2) & u(1) \\ \vdots & \vdots \\ u(N) & u(N-1) \end{bmatrix}$

is λ times the second column.

Those two columns are thus linearly dependent, and a least squares estimate cannot be calculated.

Question 16

1 / 1 pts

Consider the following dynamical system:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t)x_2(t) + u(t)x_1^2(t) \\ \cos\left(\frac{\pi}{2}x_1(t)\right) \end{bmatrix}$$

For $u(t) = 1$, consider the following equilibrium point $\vec{x}^* = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Let $\vec{\tilde{x}}(t) = \vec{x}(t) - \vec{x}^*$ and $\tilde{u}(t) = u(t) - 1$. We wish to write a system as

$$\frac{d}{dt}\vec{\tilde{x}}(t) = A\vec{\tilde{x}}(t) + B\tilde{u}(t)$$

Which of the following is a correct linearization:

- (selected) $A = \begin{bmatrix} 1 & 1 \\ -\frac{\pi}{2} & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- $A = \begin{bmatrix} x_2(t) & x_1(t) \\ -\frac{\pi}{2}\sin\left(\frac{\pi}{2}x_1(t)\right) & 0 \end{bmatrix}, B = \begin{bmatrix} x_1^2(t) \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} x_2(t) + 2u(t)x_1(t) & x_1(t) \\ -\frac{\pi}{2}\sin\left(\frac{\pi}{2}x_1(t)\right) & 0 \end{bmatrix}$, $B = \begin{bmatrix} x_1^2(t) \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -\frac{\pi}{2} \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 0 & \frac{\pi}{2} \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ -\frac{\pi}{2} & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is correct.

The respective Jacobians are

$$A = \begin{bmatrix} x_2(t) + 2u(t)x_1(t) & x_1(t) \\ -\frac{\pi}{2}\sin\left(\frac{\pi}{2}x_1(t)\right) & 0 \end{bmatrix}, B = \begin{bmatrix} x_1^2(t) \\ 0 \end{bmatrix}$$

and they need to be evaluated at the equilibrium points.

Question 17

1 / 1 pts

Which of the following is a valid SVD for $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$?

$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\sigma_1 = 1,\ \sigma_2 = 1$

$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\sigma_1 = 1,\ \sigma_2 = 1$

$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\sigma_1 = 1,\ \sigma_2 = -1$

$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $\sigma_1 = 0.5,\ \sigma_2 = 0.5$

$\vec{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $\sigma_1 = 1,\ \sigma_2 = 1$

Let's multiply out,

$$U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

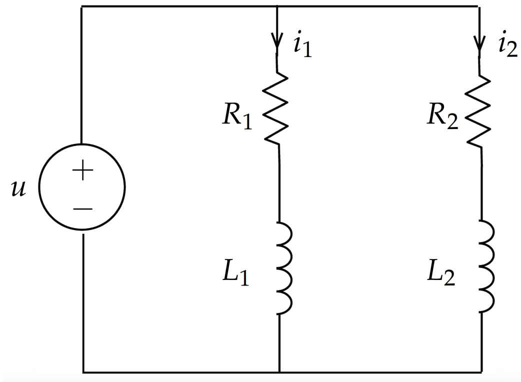
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Question 18

1 / 1 pts

Consider the circuit below, where $u(t)$ is the input and $i_1(t)$and $i_2(t)$ are the state variables:



Suppose, $R_1 = 1$ mΩ, $L_1 = 1$ mH , $L_2 = 2$ mH. For which value of R_2 is this system uncontrollable?

- $\mathrm{R}_2 = 1\ \mathrm{m}\Omega$
- $\mathrm{R}_2 = 0\ \Omega$
- (selected) $\mathrm{R}_2 = 2\ \mathrm{m}\Omega$
- None. It is controllable for all values of R_2.
- $\mathrm{R}_2 = 0.5\ \mathrm{m}\Omega$

Here, $\frac{R_1}{L_1} = \frac{R_2}{L_2}$.

As, R_1 = 1 mΩ, L_1 = 1 mH, and L_2 = 2 mH. So, R_2 = 1 mΩ.

Using KVL, $u = R_1 i_1 + L_1 \frac{di_i}{dt} = R_2 i_2 + L_2 \frac{di_2}{dt}$.

It follows,

$\frac{di_1}{dt} = -\frac{R_1}{L_1} i_1 + \frac{1}{L_1} u$, and

$\frac{di_2}{dt} = -\frac{R_2}{L_2} i_2 + \frac{1}{L_2} u$.

So,

$$\frac{d}{dt}\vec{i} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} \end{bmatrix} \vec{i} + \begin{bmatrix} \frac{1}{L_1} \frac{1}{L_2} \end{bmatrix} u.$$

So, $A = \begin{bmatrix} -\frac{R_1}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} \end{bmatrix}$ and B = $\begin{bmatrix} \frac{1}{L_1} \\ \frac{1}{L_2} \end{bmatrix}$

There are 2 state vectors. Controllability matrix, $C = [B \; AB]$.

$$C = \begin{bmatrix} \frac{1}{L_1} & -\frac{R_1}{L_1^2} \\ \frac{1}{L_2} & -\frac{R_2}{L_2^2} \end{bmatrix}.$$

For matrix C to have rank <2, we need, ratio of the matrix elements in each column equal.

So, $\frac{R_1}{L_1} = \frac{R_2}{L_2}$.

Question 19

1 / 1 pts

Let A be an $m \times n$ real matrix with SVD in standard outer product form

$A = \sigma_1 \vec{u}_1 \vec{v}_1^\top + \sigma_2 \vec{u}_2 \vec{v}_2^\top + \sigma_3 \vec{u}_3 \vec{v}_3^\top$ with $\sigma_1 \geq \sigma_2 \geq \sigma_3 > 0$.

Which of the following is NOT true:

- $A^\top A \, \vec{v}_2 = \sigma_2^2 \vec{v}_2$
- $n \geq 3$
- $\text{rank}(A^\top) = 3$
- $\vec{v}_1 \vec{v}_1^\top = 1$
- $[\, \vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3 \,]^\top [\, \vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3 \,] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Question 20

1 / 1 pts

Consider the system:

$$\frac{dx(t)}{dt} = (a - by(t))x(t)$$
$$\frac{dy(t)}{dt} = (cx(t) - d)y(t)$$

where, $x(t)$ and $y(t)$ are non-negative state variables and a, b, c, and d are positive constants. Professor Arcak linearized this model around one of its equilibrium points (he won't tell you which) and found that the resulting matrix A has complex eigenvalues. What are these eigenvalues?

- (selected) $\lambda_{1,2} = \pm j\sqrt{ad}$
- $\lambda_{1,2} = -bd/c \pm jac/b$
- $\lambda_{1,2} = a \pm jd\sqrt{b/c}$
- $\lambda_{1,2} = -d \pm ja$
- $\lambda_{1,2} = -d \pm ja\sqrt{c/b}$

For equilibrium,

$\frac{dx}{dt} = (a - by^*)x^* = 0$ and

$\frac{dx}{dt} = (cx - d^*)y^* = 0$

So, the two equilibrium conditions are,

$x^* = 0, y^* = 0$ and $x^* = \frac{d}{c}, y^* = \frac{a}{b}$.

The Jacobian matrix for linearization corresponding to the equilibrium, $x^* = 0, y^* = 0$ is,

$\begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}$. So,

$\lambda = a, -d$.

Similarly, for $x^* = \frac{d}{c}, y^* = \frac{a}{b}$, Jacobian,

$J = \begin{bmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{bmatrix}$. Solving the characteristic equation,

$\lambda = \pm \mathrm{j}\sqrt{ad}$

Question 21

1 / 1 pts

A linear dynamical system is given below:

$$\frac{d}{dt}\vec{x} = \mathbf{A}\vec{x} + \mathbf{B}\vec{u}$$

The input $\vec{u}$ is a constant. What property of the matrix $\mathbf{A}$ is required so that the system has exactly two distinct equilibrium points?

- Always possible
- Not possible

The following equation must be satisfied if x^* is an equilibrium point:

$$\vec{0} = \mathbf{A}\vec{x}^* + \mathbf{B}\vec{u}$$

$$\mathbf{A}\vec{x}^* = -\mathbf{B}\vec{u}$$

Solving for x^* is solving a linear system, which cannot have exactly two distinct solutions.

- $\mathbf{B}\vec{u}$ is in the column space of $\mathbf{A}$
- The system is controllable
- $\mathbf{A}$ is not invertible

Question 22

1 / 1 pts

An invertible $n \times n$ matrix A has n distinct non-zero singular values. How many singular value decompositions $A = U\Sigma V^\top$ does A have?

- 2^{n-1}

- ○ n^2
- ○ $n!$
- ◉ 2^n
- ○ Not enough information to determine

Question 23

1 / 1 pts

Which of the following could be a non-zero singular value for matrix B below?

$$B = \begin{bmatrix} 1 & 5 & 1 & 1 & 2 \\ 2 & 7 & 2 & 9 & 4 \\ 3 & 3 & 3 & 4 & 6 \end{bmatrix}$$

- ○ 1.01+2.14j
- ○ -1.05
- ○ 1.01-2.14j
- ○ -100
- ◉ 4.04

The question is asking for non-zero singular value. We know for an SVD for a real matrix the singular value cannot be complex or negative. So, the only answer we are left with is the positive real number.

Question 24

1 / 1 pts

A discrete-time system is modeled by the following equation:
$x(t+1) = ax(t) + bu(t) + e(t)$, where $e(t)$ is the system disturbance. The inputs and outputs at different time steps are :
$x(0) = 1,\ x(1) = 2,\ x(2) = 1,\ x(3) = -2,\ u(0) = 1,\ u(1) = 0, u(2) = 1.$
What are the least-squares estimates of the parameters a and b?

- ○ $a = \frac{1}{2}$ and $b = 1$
- ◉ $a = \frac{1}{2}$ and $b = -\frac{1}{2}$
- ○ $a = 1$ and $b = -\frac{1}{2}$
- ○ $a = 1$ and $b = 1$
- ○ $a = 1$ and $b = -1$

$$D^T D = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 2 \end{bmatrix}$$

$$(D^T D)^{-1} = \frac{1}{12-4} \begin{bmatrix} 2 & -2 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

So,

$$\vec{\hat{p}} = (D^T D)^{-1} D^T y = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Using the given conditions,

$2 \; = \; a + b$

$1 = 2a$

$-2 = a + b$

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} e(0) \\ e(1) \\ e(2) \end{bmatrix}$$

Which can be represented as,

$\vec{y} = D\vec{p} + \vec{e}$, where,

$$\vec{y} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}, \vec{p} = \begin{bmatrix} a \\ b \end{bmatrix}, \vec{e} = \begin{bmatrix} e(0) \\ e(1) \\ e(2) \end{bmatrix}.$$

The least-square estimate for p,

$\vec{\hat{p}} = (D^T D)^{-1} D^T \vec{y}$.

Question 25

1 / 1 pts

Consider the continuous-time system

$$\frac{dx_1(t)}{dt} = x_2(t)$$
$$\frac{dx_2(t)}{dt} = u(t)$$

where $u(t)$ is the input. Professor Sanders discretized this model with a sampling period T and obtained,

$$\vec{x_d}(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x_d}(k) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u_d(k).$$

What is the sampling period, T Professor Sanders used?

- $T = 0.5$
- $T = 1$ (selected)
- $\mathrm{T} = 1/\sqrt{2}$
- $T = 0.1$
- $T = 0.2$

We found,

$$\begin{bmatrix} x_1(t+T) \\ x_2(t+T) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u(t)$$, which is true if $T = 1$.

Calculating the change in x_1 and x_2 in T,

$$x_2(t+T) - x_2(t) = \int_t^{t+T} u(\tau)d\tau = Tu(t)$$

$$x_1(t+T) - x_1(t) = \int_t^{t+T} x_2(\tau)d\tau$$

$$= \int_t^{t+T} \left[x_2(t) + (\tau - t)u(t)\right] d\tau$$

$$= \int_t^{t+T} x_2(t)d\tau + \int_t^{t+T} (\tau - t)u(t)d\tau$$

$$= Tx_2(t) + \frac{T^2}{2}u(t)$$

So,

$$x_1(t+T) = x_1(t) + Tx_2(t) + \frac{T^2}{2}u(t)$$

$$x_2(t+T) = x_2(t) + Tu(t)$$

In matrix form,

$$\begin{bmatrix} x_1(t+T) \\ x_2(t+T) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u(t)$$

Quiz Score: **25** out of 25