Math 54 First Midterm Exam, Prof. Srivastava
February 20, 2020, 5:10pm–6:30pm, 150 Wheeler.

Name: ______________________________

SID: ______________________________

Instructions: Write all answers in the provided space. This exam includes two pages of scratch paper, which must be submitted but will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious.

Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed.

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Sign here: ______________________________

Question	Points
1	20
2	20
3	13
4	6
5	13
6	13
7	15
Total:	100

Do not turn over this page until your instructor tells you to do so.

Name and SID: ______________________________

1. (20 points) Circle always true (**T**) or sometimes false (**F**) for each of the following. There is no need to provide an explanation. Two points each.

   (a) If a linear system has strictly more equations than variables it must be inconsistent. **T** **F**

   (b) If A is an $m \times n$ matrix such that $A\mathbf{x} = 0$ has only the trivial solution, then $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $b \in \mathbb{R}^m$. **T** **F**

   (c) If R is the RREF of A and the columns of R are linearly dependent, then the columns of A must be linearly dependent. **T** **F**

   (d) If $\mathbf{v}, \mathbf{w}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^4$ are vectors such that $\mathbf{v} \in \operatorname{span}\{\mathbf{w}, \mathbf{a}\}$ and $\mathbf{w} \in \operatorname{span}\{\mathbf{a}, \mathbf{b}\}$ then it must be the case that $\mathbf{v} \in \operatorname{span}\{\mathbf{a}, \mathbf{b}\}$. **T** **F**

   (e) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^4$ are vectors such that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ must be linearly dependent. **T** **F**

   (f) If A is an $n \times n$ matrix with $\operatorname{Nul}(A) = \{0\}$, then $\det(A) \neq 0$. **T** **F**

   (g) If $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathbb{R}^n$ and the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one to one, then $\operatorname{span}\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\} = \mathbb{R}^m$. **T** **F**

   (h) If $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly independent and $T : \mathbb{R}^n \to \mathbb{R}^m$ is one to one, then $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)$ must be linearly independent. **T** **F**

   (i) The set $\{\mathbf{x} \in \mathbb{R}^3 : x_1 - x_2 = x_3\}$ is a subspace of $\mathbb{R}^3$. **T** **F**

   (j) If C and A are square matrices such that $CA = I$, then $AC = I$. **T** **F**

2. Give an example of each of the following, explaining why it has the required property, or explain why no such example exists.

   (a) (4 points) A basis of $\mathbb{R}^3$ containing both of the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$.

   (b) (4 points) A vector $\mathbf{x} \in \mathbb{R}^2$ whose coordinate vector relative to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 9 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

   (c) (4 points) A 2×3 matrix A with $\mathrm{Col}(A) = \mathbb{R}^2$ and $\mathrm{Nul}(A) = \mathbb{R}^3$.

[Scratch Paper 1]

(d) (4 points) A 2×3 matrix with $\mathrm{Nul}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(e) (4 points) A 3×3 matrix A whose inverse is equal to $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

3. Consider the matrix
$$A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 1 \\ 3 & 8 & 3 \end{bmatrix}.$$

(a) (5 points) Compute the determinant of A and explain why A is invertible.

(b) (5 points) Compute A^{-1}.

(c) (3 points) Use your answer to (b) to solve $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix}$.

4. (6 points) State precisely the definition of a *one to one* linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

5. (13 points) Consider the following subspaces of $\mathbb{R}^3$:

$$H_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, H_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Find a nonzero vector $\mathbf{v} \in \mathbb{R}^3$ which belongs to *both* subspaces, i.e., $\mathbf{v} \in H_1 \cap H_2$.

6. (13 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} s \\ 7 \end{bmatrix}.$$

Find $T(\mathbf{e}_2)$, where $\mathbf{e}_2$ is the second standard basis vector. For which values of s is T onto?

7. Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by:

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$

and let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the geometric linear transformation which reflects a vector $\mathbf{x} \in \mathbb{R}^2$ across the line $x_1 = -x_2$.

(a) (5 points) Show that T_2 is invertible, and describe the inverse transformation $T_2^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$.

(b) (5 points) Find the standard matrix of the composition $T = T_1 \circ T_2^{-1} : \mathbb{R}^2 \to \mathbb{R}^3$.

(c) (5 points) Is there a vector $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$? If so, find it, if not, explain why.

[Scratch Space 2]