MATH 1B MIDTERM 1 (002)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Formulae

$$\int \tan(x)\,dx = \ln|\sec(x)| + C \qquad \int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$

$$\int \frac{1}{1+x^2}dx = \arctan(x) + C \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$$

$$\frac{d\tan(x)}{dx} = \sec^2(x) \qquad \frac{d\sec(x)}{dx} = \tan(x)\sec(x)$$

$$1 = \sin^2(x) + \cos^2(x) \qquad 1 + \tan^2(x) = \sec^2(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$|E_{T_n}| \leq \frac{K(b-a)^3}{12n^2} \qquad |E_{S_n}| \leq \frac{K(b-a)^5}{180n^4}$$

Name: [redacted]

Student ID: [redacted]

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This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Compute the following integrals:

(a) (10 points)

$$\int x\sin(x)\,dx$$

Solution: $u = x \quad du = dx$

$dv = \sin x\,dx \quad v = -\cos x$

$$\int x\sin x\,dx = -x\cos x - \int -\cos x\,dx$$

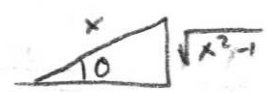
$$= -x\cos x + \int \cos x\,dx$$

$$= \boxed{-x\cos x + \sin x + C}$$

(b) (15 points)

$$\int \frac{\sqrt{x^2-1}}{x^4}\,dx$$

Solution: $x = \sec\theta \quad dx = \sec\theta\tan\theta\,d\theta$



$\sin\theta = \frac{\sqrt{x^2-1}}{x}$

$$\int \frac{\sqrt{\sec^2\theta-1}}{\sec^4\theta}\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\tan\theta}{\sec^4\theta}\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}\cos^3\theta\,d\theta$$

$$= \int \sin^2\theta\cos\theta\,d\theta \qquad u = \sin\theta$$

$$du = \cos\theta\,d\theta$$

$$= \int u^2\,du = \frac{1}{3}u^3 = \frac{1}{3}\sin^3\theta + C = \boxed{\frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C}$$

2. (a) (15 points) Express the following rational function

$$\frac{3x^3 + 2x^2 + x + 1}{x^4 + x^2}$$

as a sum of partial fractions.

Solution:

$$\frac{3x^3 + 2x^2 + x + 1}{x^2(x^2+1)} = \frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{x^2+1}$$

$$3x^3 + 2x^2 + x + 1 = A(x^2+1) + B(x)(x^2+1) + (Cx+D)x^2$$

$$= Ax^2 + A + Bx^3 + Bx + Cx^3 + Dx^2$$

$$= (B+C)x^3 + (A+D)x^2 + Bx + A$$

$A = 1 \quad B = 1 \quad A + D = 2 \quad B + C = 3$

$D = 1 \quad C = 2$

$$\boxed{\frac{1}{x^2} + \frac{1}{x} + \frac{2x+1}{x^2+1}}$$

(b) (10 points) Hence evaluate the integral

$$\int \frac{3x^3 + 2x^2 + x + 1}{x^4 + x^2}\, dx$$

Solution:

$$\int \frac{1}{x^2} + \frac{1}{x} + \frac{2x+1}{x^2+1}\, dx$$

$$= \int \frac{1}{x^2}\, dx + \int \frac{1}{x}\, dx + \int \frac{2x}{x^2+1}\, dx + \int \frac{1}{x^2+1}\, dx$$

$u = x^2 + 1 \quad du = 2x\,dx$

$$= \boxed{-\frac{1}{x} + \ln|x| + \ln|x^2+1| + \arctan(x) + C}$$

3. (25 points) Find the arc length of the curve

$$y = \frac{x^2}{4} - \ln(\sqrt{x}) = \frac{x^2}{4} - \frac{1}{2}\ln x$$

between $x = 1$ and $x = 2$.

Solution:

$$\ell = \int_a^b \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx$$

$$\frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{4} - 2\left(\frac{x}{2}\right)\left(\frac{1}{2x}\right) + \frac{1}{4x^2}$$

$$= \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}$$

$$\left(\frac{dy}{dx}\right)^2 + 1 = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}$$

$$= \left(\frac{x}{2} + \frac{1}{2x}\right)^2$$

$$\ell = \int_1^2 \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\, dx$$

$$= \int_1^2 \left(\frac{x}{2} + \frac{1}{2x}\right) dx$$

$$= \frac{x^2}{4} + \frac{1}{2}\ln|x| \Big|_1^2$$

$$= \frac{4-1}{4} + \frac{1}{2}(\ln 2 - \ln 1)$$

$$= \boxed{\frac{3}{4} + \frac{\ln 2}{2}}$$

PLEASE TURN OVER

4. Determine if the following improper integrals are convergent or divergent. Justify your answers.

(a) (10 points)

$$\int_{-\infty}^{-1} \frac{\sin(x^2)+2}{x^2}\,dx$$

Solution:

$$\lim_{a \to -\infty} \int_a^{-1} \frac{\sin(x^2)}{x^2} + \frac{2}{x^2}\,dx$$

because $\sin(x^2) \in [-1,1]$, $\frac{-1}{x^2} \le \frac{\sin(x^2)}{x^2} \le \frac{1}{x^2}$ $\forall x \in (-\infty,-1)$

$\frac{1}{x^2} \le \frac{\sin(x^2)+2}{x^2} \le \frac{3}{x^2}$ $\forall x \in (-\infty,-1)$

$\therefore \lim_{a\to-\infty} \int_a^{-1} \frac{1}{x^2}\,dx \le \lim_{a\to-\infty} \int_a^{-1} \frac{\sin(x^2)+2}{x^2}\,dx \le \lim_{a\to-\infty} \int_a^{-1} \frac{3}{x^2}\,dx$

by sandwich thm, $\int_{-\infty}^{-1} \frac{\sin(x^2)+2}{x^2}\,dx$ $\boxed{\text{converges.}}$

$\lim_{a\to-\infty} -\frac{1}{x}\Big|_a^{-1} = -\lim_{a\to-\infty}\left(-1-\frac{1}{a}\right) = 1$ converges

$\lim_{a\to-\infty} \frac{-3}{x}\Big|_a^{-1} = -3\lim_{a\to-\infty}\left(-1-\frac{1}{a}\right) = 3$

(b) (15 points)

$$\int_0^1 \frac{\cos(1/x)}{x^2}\,dx$$

$\lim_{a\to0^+} \int_a^1 \frac{\cos(1/x)}{x^2}\,dx$ $\quad u = \frac{1}{x}$ $\quad du = -\frac{1}{x^2}\,dx$

$\int \frac{\cos(1/x)}{x^2}\,dx = -\int \cos u\,du = -\sin u = -\sin\left(\frac{1}{x}\right)$

$\lim_{a\to0^+} \int_a^1 \frac{\cos(1/x)}{x^2}\,dx = \lim_{a\to0^+} -\sin\left(\frac{1}{x}\right)\Big|_a^1$

$= -\lim_{a\to0^+}\left(\sin(1) - \sin\left(\frac{1}{a}\right)\right)$ $\quad$ let $t = \frac{1}{a}$

$= -\sin(1) + \underbrace{\lim_{t\to\infty} \sin(t)}_{\text{DNE}}$

$\boxed{\text{diverges}}$

5. (25 points) For n a positive integer, let T_n be the trapezoidal approximation of the definite integral

$$\int_{-1}^{1} x^4 - 6x^2 + x + \frac{23}{10}\,dx.$$

Is it possible that $T_{100} = 0.99$? Carefully justify your answer. Hint: What is the exact value of the integral?

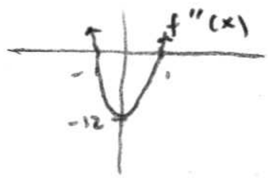
Solution:

$f(x) = x^4 - 6x^2 + x + \frac{23}{10}$

$E_T(n) \le \frac{K(b-a)^3}{12n^2}$

$f'(x) = 4x^3 - 12x + 1$

$f''(x) = 12x^2 - 12$



$K = \max \left| f''(x) \right|_{x \in [-1,1]}$

max when $x = 0$

$K = 12$

$|E_T(100)| \le \frac{12(2)^3}{12(100)^2}$

$|E_T(100)| \le 8 * 10^{-4}$

$$\int_{-1}^{1} x^4 - 6x^2 + x + \frac{23}{10}\,dx = \left.\underbrace{\tfrac{1}{5}x^5}_{\text{odd}} \underbrace{- 2x^3}_{\text{odd}} + \underbrace{\tfrac{1}{2}x^2}_{\text{even}} + \underbrace{\tfrac{23}{10}x}_{\text{odd}}\right|_{-1}^{1}$$

$= \frac{1}{5}(2) - 2(2) + \frac{23}{10}(2)$

$= \frac{2}{5} + \frac{23}{5} - 4$

$= 5 - 4$

$= 1$

$0.1 > \overset{\text{max error}}{E_T(100)}$

$\boxed{\text{no, not possible}}$

END OF EXAM