SPRING 2020 MATH 54 MIDTERM 2 SOLUTIONS

Q1 True False.

(a) True. Take A with rows equal to any basis of $W^\perp$.
(b) True. Take A to be the matrix of the orthogonal projection onto W.
(c) False. Check that the matrices $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in H$ but their sum is not.
(d) True. Each of S, T, S^{-1} is invertible and a composition of invertible transformations is invertible, so onto.
(e) False. Consider the identity.
(f) False. If $u \in \mathbb{R}^2$ is any nonzero vector, $u^T u \neq 0$ is invertible, but uu^T is rank one, so not invertible.
(g) True. If $A = PBP^{-1}$ then $x \in \mathrm{Nul}(A) \iff P^{-1}x \in \mathrm{Nul}(B)$, so they have the same nullity, and by rank nullity the same rank.
(h) False. Every nonzero $1 \times n$ matrix A is row equivalent to e_1^T, but not every vector in $\mathbb{R}^n$ has the same distance from e_1 and $\mathrm{Row}(A)$.
(i) True. $z \in \mathrm{Row}(A)^\perp$, but since $\mathrm{Row}(A) = \mathbb{R}^2$ we must have $z = 0$.
(j) False. The question is asking whether $(P^{-1}x)^T(P^{-1}y) = x^T(P^{-1})TP^{-1}y = x^Ty$ for every x, y. This is false unless $(P^{-1})^T P^{-1} = I$, of which there are many examples.

Q3 Examples

(a) Any 2×2 matrix with rank one will do, since by rank nullity it also has nullity 1, and any two vector spaces of the same dimension are isomorphic. For example $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
(b) Does not exist. If $A = PDP^{-1}$ then $A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$ is also diagonalizable.
(c) For any basis B it is true that $[b_1]_B = e_1$ since we have the unique linear combination $b_1 = 1b_1 + 0b_2$. Thus, the desired basis has $b_2 = e_1$. Taking b_1 to be any unit vector orthogonal to this (i.e., $\pm e_2$) yields two possible bases with this property.
(d) Does not exist. Since $W \neq \mathbb{R}^3$ the kernel of proj_W is equal to $W^\perp$, which is nontrivial. Thus proj_W cannot be one to one.

Math 54 MT2 Q4

Let $\beta = \{1, t, t^2\}$ be std. basis of $\mathbb{P}^2$ polynomials, and so any polynomial $\in \mathbb{P}^2$ has the form $p = a_0 + a_1 t + a_2 t^2$. Since this is a linear comb of std basis vectors, $[p]_\beta = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$. This means that $p \mapsto [p]_\beta$ is an isomorphism from $\mathbb{P}^2$ ~~onto~~ $\mathbb{R}^3$, meaning vector operations in $\mathbb{P}^2$ correspond to operations in $\mathbb{R}^3$. This way, we can consider T as a composite transformation of T_1 then T_2, where T_1 simply maps a vector in $\mathbb{P}^2$ to its corresponding representation in $\mathbb{R}^3$, and where T_2 is the transformation from $[p]_\beta$ to $\begin{bmatrix} p(0) \\ p'(0) \\ p''(0) \end{bmatrix}$.

To show that T is invertible, it suffices to show that both T_1 & T_2 are separately invertible. T_1's invertibility is trivial as it is an isomorphism (can always switch back and forth between representations). Next we show T_2's invertibility.

Since $p(t) = a_0 + a_1 t + a_2 t^2$, $p'(t) = a_1 + 2a_2 t$, $p''(t) = 2a_2$ from calculus,

$p(0) = a_0$, $p'(0) = a_1$, $p''(0) = 2a_2$.

Therefore, $T_2\left(\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} a_0 \\ a_1 \\ 2a_2 \end{bmatrix}$. We can find a std. matrix for T_2 by noticing that T_2 simply multiplies the last entry by 2. Then, $A_2 \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ 2a_2 \end{bmatrix} \Rightarrow A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. A_2 keeps all entries intact except scaling last entry by 2. A transformation is invertible iff its std matrix is invertible. A_2 is diagonal and so its det is $(1)(1)(2) \neq 0 \Rightarrow A_2$ is invertible $\Rightarrow T_2$ is invertible. To find its inverse, we note that $T_2^{-1}\left(\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}\right)$ should give $\begin{bmatrix} a_0 \\ a_1 \\ a_2/2 \end{bmatrix}$, scaling the last term by 1/2 instead of 2. Then, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$. Checking, we conclude that $AA^{-1} = I$ so A_2^{-1} is std. matrix of T_2^{-1}.

Since $T(p) = T_2(T_1(p))$, $T^{-1}(q) = T_1^{-1}(T_2^{-1}(q))$. This means, given a vector $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = q \in \mathbb{R}^3$, we first half the last term, producing $\begin{bmatrix} a_0 \\ a_1 \\ a_2/2 \end{bmatrix}$, and then represent it as a polynomial, $a_0 + a_1 t + \frac{1}{2} a_2 t^2 = p \in \mathbb{P}^2$. ($T^{-1}$ acts on a vector $q \in \mathbb{R}^3$ and returns a vector $p \in \mathbb{P}^2$)

Math 54 MT2 Q5

Let $X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2\times2}$ and $X^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$. Since M_{2x2} is isomorphic to $\mathbb{R}^4$, we work in $\mathbb{R}^4$ (with $[X]_\beta = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}$)

Then, S represents a transformation from $\begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}$ to $\begin{bmatrix} \bullet \\ a_{12}-a_{21} \\ a_{21}-a_{12} \\ \bullet \end{bmatrix}$, since $X - X^T = \begin{bmatrix} 0 & a_{12}-a_{21} \\ a_{21}-a_{12} & 0 \end{bmatrix}$. The std. matrix for S can be found by inspection

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{(std matrix}} \underbrace{\begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}}_{\text{input}} = \underbrace{\begin{bmatrix} 0 \\ a_{12} \ a_{21} \\ a_{21}-a_{12} \\ 0 \end{bmatrix}}_{\text{transformed output.)}}$$

From here, we wish to diagonalize this std matrix, denoted from here as $[S]_E$, by finding eigenvalues & eigenvectors

$$\left|[S]_E - \lambda I\right| = \begin{vmatrix} -\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & -1 & 0 \\ 0 & -1 & 1-\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda \cdot -\lambda \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix}$$

$$= \lambda^2(\lambda^2 - 2\lambda + 1 - 1) - \lambda^2(\lambda^2 - 2\lambda) = \lambda^3(\lambda - 2) = 0 \text{ iff } \lambda_1 = 0 \text{ or } \lambda_2 = 2.$$

$\lambda_1 = 0$: $[S]_E - \lambda I = 0$: $\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$: $x_2 = x_3$, x_1, x_3, x_4 free

$\lambda_2 = 2$: $[S]_E - \lambda I = 0$: $\left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$: $x_2 = -x_3$; x_3 free; $x_1 = 0$; $x_4 = 0$

For $\underset{=0}{\lambda_1}$, eigenspace is $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$. For $\underset{=2}{\lambda_2}$, eigenspace is $\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$.

Therefore, $[S]_E = PDP^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} P^{-1}$ Translating these basis vectors (columns of P) in $\mathbb{R}^4$ into a basis for M_{2x2}, we get a basis (by the diagonal matrix representation) of: $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\}$, and ${}_\beta[S]_\beta = D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

To find the kernel of S, we observe that in the diagonal matrix representation, three of the eigen "matrices" corresponded to eigenvalue $\lambda_1 = 0$. This means that any matrix $\in M_{2x2}$ within the eigenspace of $\lambda_1 = 0$ gets mapped to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. The three corresponding matrices to $\lambda_1 = 0$ are $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$, which together form a basis for Ker(S).

Math 54 MT2 Q6

a) Note that because A is upper triangular, its eigenvalues, $\lambda_1=1, \lambda_2=-1, \lambda_3=2$, lie on its diagonal. Since there are 3 distinct eigenvalues for this 3x3 matrix, A is diagonalizable : $A=PDP^{-1}$ for some invertible P and diagonal D. The strategy for this problem uses the fact that $A^k = PD^kP^{-1}$ ($A^2=(PDP^{-1}PDP^{-1})=PD^2P^{-1}$, and so on)

To diagonalize A, we find its eigenvectors corresponding to $\lambda_1, \lambda_2, \lambda_3$.

$\underset{1}{\lambda_1}$: $A-\lambda I=0$: $\left[\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_1$ free, $x_2=x_3=0$

$\underset{-1}{\lambda_2}$: $A-\lambda I=0$: $\left[\begin{array}{ccc|c} 2 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_2$ free, $x_1=-x_2$, $x_3=0$

$\underset{2}{\lambda_3}$: $A-\lambda I=0$: $\left[\begin{array}{ccc|c} -1 & 2 & 3 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & -4/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -17/3 & 0 \\ 0 & 1 & -4/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_3$ free, $x_1 = 17/3\, x_3$, $x_2 = 4/3\, x_3$

Then, the eigenvectors corresponding to $\lambda_1=1, \lambda_2=-1, \lambda_3=2$, are $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}$, and $\begin{bmatrix}17\\4\\3\end{bmatrix}$, respectively. Thus, $A=PDP^{-1}=$

$= \begin{bmatrix}1&1&17\\0&-1&4\\0&0&3\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&2\end{bmatrix}\begin{bmatrix}1&1&-7\\0&-1&4/3\\0&0&1/3\end{bmatrix}$, where P^{-1} was obtained by row-reducing $[P \mid I]$. Using $A^k = PD^kP^{-1}$ and that D^k is just a matrix with the entries on the diagonal raised to the k^{th} power,

$A^{99} = \begin{bmatrix}1&1&17\\0&-1&4\\0&0&3\end{bmatrix}\begin{bmatrix}1^{99}&0&0\\0&-1^{99}&0\\0&0&2^{99}\end{bmatrix}\begin{bmatrix}1&1&-7\\0&-1&4/3\\0&0&1/3\end{bmatrix} = \begin{bmatrix}1&-1&17\,2^{99}\\0&1&\text{some \#}\\0&0&\text{some other \#}\end{bmatrix}\begin{bmatrix}1&1&-7\\0&-1&4/3\\0&0&1/3\end{bmatrix}$. We see that because the third entry of the second col of P^{-1} is 0 and that the second col of A^{99} is a linear combination of columns of (PD^{99}) with weights as the second col of P^{-1}, the second col of $A^{99} = 1\begin{bmatrix}1\\0\\0\end{bmatrix} - 1\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}2\\-1\\0\end{bmatrix}$.

b) Note that A is invertible because it is the product of invertible matrices : P defined to be invertible but also because it is a matrix with lin ind. eigenvectors as columns, and D because $\det(D)=\lambda_1\lambda_2\lambda_3 \neq 0$. Note that because $A=PDP^{-1}$ and each of P, D, P^{-1} are invertible, $A^{-1}=(PDP^{-1})^{-1} = P^{-1}D^{-1}P$. Since $D=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&2\end{bmatrix}$ and corresponding eigenvalues of A^{-1} are $\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}$ respectively, $D^{-1}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1/2\end{bmatrix}$. Then, $A^{-99}=(A^{-1})^{99} = P^{-1}(D^{-1})^{99}P = \begin{bmatrix}1&1&-7\\0&-1&4/3\\0&0&1/3\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&(1/2)^{99}\end{bmatrix}\begin{bmatrix}1&1&17\\0&-1&4\\0&0&3\end{bmatrix}$

$= \begin{bmatrix}1&-1&\#\\0&1&\#\\0&0&\#\end{bmatrix}\begin{bmatrix}1&1&17\\0&-1&4\\0&0&3\end{bmatrix}$ Using similar logic to (a) in finding A^{99}'s second col, we see that second col of $A^{-99} = 1\begin{bmatrix}1\\0\\0\end{bmatrix} - 1\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}2\\-1\\0\end{bmatrix}$.

Thus, since second col of $(A^{99}-A^{-99})$ = second col of A^{99} - second col of A^{-99}, second col of $(C=A^{99}-A^{-99}) = \begin{bmatrix}2\\-1\\0\end{bmatrix} - \begin{bmatrix}2\\-1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

Math 54 MT2 Q7

$A = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & 2 & 1 \end{bmatrix}$. To find Nul A, we solve $Ax = 0$:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & -1 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -2 & -1 & 0 \end{array}\right] \Rightarrow \begin{array}{ll} x_1 = x_3 + x_4 & x_3 \text{ free} \\ x_2 = 2x_3 + x_4 & x_4 \text{ free.} \end{array}$$

A basis for Nul $A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. From this basis, we find an orthogonal basis by Gram-Schmidt:

Let $b_1 = v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$. Then, $b_2 = v_2 - \text{proj}_{b_1} v_2 = v_2 - \frac{v_2 \cdot b_1}{b_1 \cdot b_1} b_1$

$= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/2 \\ -1 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 1 \end{bmatrix}$. Then, $\{b_1, b_2\}$ form an orthogonal basis of Nul A. Using this basis, we proceed with orthogonal decomp:

To find $\hat{b} \in$ Nul A closest to b, we solve for $\text{proj}_{\text{Nul}A}\, b = \hat{b}$, which is the closest vector to b in Nul A by best approx theorem.

$\text{proj}_{\text{Nul}A}\, b = \frac{b \cdot b_1}{b_1 \cdot b_1} b_1 + \frac{b \cdot b_2}{b_2 \cdot b_2} b_2$, where $\{b_1, b_2\}$ is the above orthogonal basis for Nul A. Then, $\hat{b} = \text{proj}_{\text{Nul}A}\, b = \frac{1}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \frac{1/2}{3/2}\begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1/6 \\ 1/3 \\ 1/6 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/6 \\ 0 \\ -1/6 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ 1/3 \end{bmatrix}$. This vector $\hat{b}$ is the closest point to b that is in Nul A.

Letting $W =$ Nul A, $b = y + z$ for $y \in W$ and $z \in W^\perp$, we see that because $\hat{b}$ is the orthogonal projection of b onto $W =$ Nul A (component of b in W), $y = \hat{b} = \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ 1/3 \end{bmatrix}$. Then, we can solve for z: $z = b - y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ 0 \\ -1/3 \end{bmatrix}$.

$\therefore\ y = \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ 1/3 \end{bmatrix} = \text{proj}_{\text{Nul}A}\, b$ = closest point in Nul A to b, and $z = \begin{bmatrix} 2/3 \\ -1/3 \\ 0 \\ -1/3 \end{bmatrix}$.