Math 54 First Midterm Exam, Prof. Srivastava
February 20, 2020, 5:10pm–6:30pm, 150 Wheeler.

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Instructions: Write all answers in the provided space. This exam includes two pages of scratch paper, which must be submitted but will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious.

Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed.

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Question	Points
1	20
2	20
3	13
4	6
5	13
6	13
7	15
Total:	100

Do not turn over this page until your instructor tells you to do so.

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1. (20 points) Circle always true (**T**) or sometimes false (**F**) for each of the following. There is no need to provide an explanation. Two points each.

   (a) If a linear system has strictly more equations than variables it must be inconsistent.

   > **Solution:** False. Consider any homogeneous system $Ax = 0$, or a system whose equations are scalar multiples of one equation.

   T **F**

   (b) If A is an $m \times n$ matrix such that $A\mathbf{x} = 0$ has only the trivial solution, then $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $b \in \mathbb{R}^m$. **T** **F**

   > **Solution:** True. The solution set of $Ax = b$ is equal to any particular solution translated by the solution set of $Ax = 0$. Algebraically, if $Ax_1 = b$ and $Ax_2 = b$ then $A(x_1 - x_2) = 0$, so we must have $x_1 = x_2$.

   (c) If R is the RREF of A and the columns of R are linearly dependent, then the columns of A must be linearly dependent. **T** **F**

   > **Solution:** True. Row operations preserve the solution set of $Ax = 0$, so $\text{Nul}(A) = \text{Nul}(R)$, and if the latter is nontrivial then so is the former.

   (d) If $\mathbf{v}, \mathbf{w}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^4$ are vectors such that $\mathbf{v} \in \text{span}\{\mathbf{w}, \mathbf{a}\}$ and $\mathbf{w} \in \text{span}\{\mathbf{a}, \mathbf{b}\}$ then it must be the case that $\mathbf{v} \in \text{span}\{\mathbf{a}, \mathbf{b}\}$. **T** **F**

   > **Solution:** True. We know $\mathbf{v} = c_1\mathbf{w} + c_2\mathbf{a}$ and $\mathbf{w} = c_3\mathbf{a} + c_4\mathbf{b}$ so $\mathbf{v} = c_1(c_3\mathbf{a} + c_4\mathbf{b}) + c_2\mathbf{a}$.

   (e) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^4$ are vectors such that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ must be linearly dependent. **T** **F**

   > **Solution:** True. The hypothesis implies in particular that $\mathbf{v}_3 \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, yielding a nontrivial linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ equal to zero after rearranging all the vectors on one side.

   (f) If A is an $n \times n$ matrix with $\text{Nul}(A) = \{0\}$, then $\det(A) \neq 0$. **T** **F**

   > **Solution:** True. If $\text{Nul}(A) = \{0\}$ then the columns of A are linearly independent, which means there is a pivot in every column, which means A is invertible so $\det(A) \neq 0$. You could also reason using any of the other equivalences in the "invertible matrix theorem" in the book.

(g) If $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathbb{R}^n$ and the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one to one, then $\text{span}\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\} = \mathbb{R}^m$. **T F**

Solution: False. Consider $\mathbf{v}_1 = e_1$, which spans $\mathbb{R}^1$, and the transformation $T : \mathbb{R}^1 \to \mathbb{R}^2$ by $T(\begin{bmatrix}1\end{bmatrix}) = \begin{bmatrix}1\\0\end{bmatrix}$.

(h) If $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly independent and $T : \mathbb{R}^n \to \mathbb{R}^m$ is one to one, then $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)$ must be linearly independent. **T F**

Solution: True. Suppose $c_1, \ldots, c_k$ are coeffs satisfying $c_1T(\mathbf{v}_1)+\ldots+c_kT(\mathbf{v}_k) = 0$. By linearity this means $T(c_1\mathbf{v}_1+\ldots+c_k\mathbf{v}_k) = 0$. But since T is one to one, we must have $c_1\mathbf{v}_1+\ldots+c_k\mathbf{v}_k = 0$, which since $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent implies $c_1 = \ldots = c_k = 0$.

(i) The set $\{\mathbf{x} \in \mathbb{R}^3 : x_1 - x_2 = x_3\}$ is a subspace of $\mathbb{R}^3$. **T F**

Solution: True. This is the nullspace of the 3×1 matrix $\begin{bmatrix}1 & -1 & -1\end{bmatrix}$.

(j) If C and A are square matrices such that $CA = I$, then $AC = I$. **T F**

Solution: True. If $CA = I$, then $CAC = C$ and both C and A must be invertible. Multiplying both sides on the left by C^{-1} yields $AC = I$.

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2. Give an example of each of the following, explaining why it has the required property, or explain why no such example exists.

(a) (4 points) A basis of $\mathbb{R}^3$ containing both of the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$.

Solution: No such basis exists, because the given vectors are linearly dependent, so any set containing them must be linearly dependent, which violates one of the conditions of being a basis.

(b) (4 points) A vector $\mathbf{x} \in \mathbb{R}^2$ whose coordinate vector relative to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 9 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Solution: Such a vector is given by $\begin{bmatrix} 1 & -2 \\ 9 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 \\ 24 \end{bmatrix}$.

(c) (4 points) A 2×3 matrix A with $\mathrm{Col}(A) = \mathbb{R}^2$ and $\mathrm{Nul}(A) = \mathbb{R}^3$.

Solution: Such a matrix does not exist by the rank nullity theorem: we must have $\dim(\mathrm{Col}(A)) + \dim(\mathrm{Nul}(A)) = 3$, which is violated above.

(d) (4 points) A 2×3 matrix with $\text{Nul}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Solution: We can reverse engineer such a matrix by remembering that $\text{Nul}(A)$ is the solution set of $Ax = 0$, which is parameterized by free variables. The given subspace is $\{ \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R}\}$, so if we find a linear system with free variable x_3 and pivot variables x_1, x_2 satisfying

$$x_2 = 0 \quad x_1 - x_3 = 0,$$

we are done. Such a system is given by the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

(e) (4 points) A 3×3 matrix A whose inverse is equal to $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

Solution: Such a matrix does not exist, because the given matrix is not invertible (the first and second columns are scalar multiples of each other), and the inverse of an invertible matrix is always invertible.

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3. Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 1 \\ 3 & 8 & 3 \end{bmatrix}.$$

$-(6-8) + 3(1-2)$

$2 - 3$

(a) (5 points) Compute the determinant of A and explain why A is invertible.

$$\det A = -1\begin{vmatrix} 2 & 1 \\ 8 & 3 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 3 & 3 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ 3 & 8 \end{vmatrix}$$

$$= -(6-8) + 3 + -6 = 2+3-6 = \boxed{-1}$$

A is invertible iff $\det A \neq 0$. Since $\det A = -1$, A is invertible.

$0 \quad 11 \quad \frac{11}{2} \quad 0 \quad \frac{11}{2} \quad 0$

(b) (5 points) Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 3 & 8 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 11 & 6 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 3 & -11/2 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & -1 & 1/2 & 0 \\ 0 & 1 & 0 & -3 & 6 & -1 \\ 0 & 0 & 1/2 & 3 & -1/2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -5 & 1 \\ 0 & 1 & 0 & -3 & 6 & -1 \\ 0 & 0 & 1 & 6 & -11 & 2 \end{array}\right]$$

$$\boxed{\begin{bmatrix} 2 & -5 & 1 \\ -3 & 6 & -1 \\ 6 & -11 & 2 \end{bmatrix}}$$

(c) (3 points) Use your answer to (b) to solve $Ax = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix}$.

$$\vec{x} = A^{-1}\begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 1 \\ -3 & 6 & -1 \\ 6 & -11 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 2-10+9 \\ -3+12-9 \\ 6-22+18 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}$$

4. (6 points) State precisely the definition of a *one to one* linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

5. (13 points) Consider the following subspaces of $\mathbb{R}^3$:

$$H_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, H_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Find a nonzero vector $\mathbf{v} \in \mathbb{R}^3$ which belongs to *both* subspaces, i.e., $\mathbf{v} \in H_1 \cap H_2$.

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6. (13 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\5\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}s\\7\end{bmatrix}.$$

Find $T(e_2)$, where e_2 is the second standard basis vector. For which values of s is T onto?

$$T(\vec{e}_1 + \vec{e}_2) = T(\vec{e}_1) + T(\vec{e}_2) \quad \text{(linearity)}$$

$$T(\vec{e}_2) = T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$$

$$= \begin{bmatrix}s\\7\end{bmatrix} - \begin{bmatrix}0\\5\end{bmatrix} = \boxed{\begin{bmatrix}s\\2\end{bmatrix}}$$

$$T(\vec{x}) = \begin{bmatrix}0 & s\\5 & 2\end{bmatrix}\vec{x}$$

T is onto if $\forall b \in \mathbb{R}^2$ $T(\vec{x}) = \vec{b}$ has at least 1 soln. For this to be true, the std. matrix of T (call this A) must have a pivot position in every row. This is true as long as

$\boxed{s \neq 0.}$

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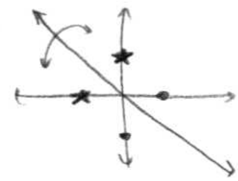
7. Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by:

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$



and let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the geometric linear transformation which reflects a vector $x \in \mathbb{R}^2$ across the line $x_1 = -x_2$.

(a) (5 points) Show that T_2 is invertible, and describe the inverse transformation $T_2^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$.

$T_2(\vec{e}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\quad T_2(\vec{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ so standard matrix of T_2 is

$A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. $\quad \det(A_2) = -1$, so A_2 is invertible and therefore T_2 is invertible.

The inverse transformation T_2^{-1} is identical to T_2. T_2^{-1} will also reflect a vector $x \in \mathbb{R}^2$ across the line $x_1 = -x_2$ and have std. matrix A_2

(b) (5 points) Find the standard matrix of the composition $T = T_1 \circ T_2^{-1} : \mathbb{R}^2 \to \mathbb{R}^3$.

$T_1\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad T_1\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ $\quad$ so $T_1(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{x}$ $\leftarrow A_1$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix}}$$

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(c) (5 points) Is there a vector $x \in \mathbb{R}^2$ such that $T(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$? If so, find it, if not, explain why.

$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 0 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & -2 & 0 \\ 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

There is no vector $\vec{x} \in \mathbb{R}^2$ such that $T(\vec{x}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

This is because $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has no solutions.

[Scratch Space]