CS 70 Discrete Mathematics and Probability Theory
Summer 2020 Amin Ghafari, Yining Liu, Khalil Sarwari

Final

- You may consult three handwritten double-sided sheets of notes. Apart from that, you may not look at books, notes, etc. Calculators, phones, computers, and other electronic devices are prohibited unless they are part of the recording submission.
- There are 9 questions on this exam, worth a total of 100 points.
- No clarification will be provided on the exam questions.
- Note that the questions vary in difficulty
- You may, without proof, use theorems and facts that were proven in the lecture, notes, discussions, and/or in homeworks unless explicitly mentioned otherwise.
- **You have 150 minutes to work on the exam. You will then have 45 minutes for scanning and uploading your answers. Late submissions will be penalized.**
- **You may use up to** $\min(x, 15)$ **minutes of the scanning time to continue working on your exam if you have lost** x **minutes due minor technical issues during the exam.**

1 (19 Points) Short Answers I

Unless otherwise stated, you must show all your work in order to get full credit.

(a) (4 points) Prove the following statement using **induction**: given $n \in \mathbb{N}$, if S is a set of cardinality n, then the power set of S has cardinality 2^n.

Answer: Base case: $P(0)$ is true, because the power set of $\emptyset$ only contains $\emptyset$.

Inductive Hypothesis: If S is a set of cardinality n, then the power set of S has cardinality 2^n.

Inductive Step: Consider a set S of cardinality $n+1$. Consider $S - \{x\}$ for some $x \in S$. Then $S - \{x\}$ is a set of cardinality n. By inductive hypothesis, the power set of $S - \{x\}$ has 2^n elements.

For each subset A of $S - \{x\}$, there are exactly two subsets of S, namely A and $A \cup \{x\}$.

Thus, the power set of S has $2^n \cdot 2 = 2^{n+1}$ elements.

(b) (3 points) For each of the following sets, **state** whether it is finite, countably infinite, or uncountable. **No need to justify/show work**.

(i) The set of prime numbers **Answer:** Countably infinite.

(ii) The set of infinite sequences of integers **Answer:** Uncountable.

(iii) The set of computer programs **Answer:** Countably infinite.

(c) (2 points) Show that if $k > 0$ is an integer, then $k+1$ is coprime to $k^2 + 2k$.

Answer: We show $\gcd(k^2+2k, k+1) = 1$.

$$\gcd(k^2+2k, k+1) = \gcd(k+1, k) = \gcd(k, 1) = \gcd(1, 0) = 1.$$

Since $\gcd(k^2+2k, k+1) = 1$, $k+1$ and k^2+2k are coprime.

***Alternative solution**:* Notice that for $x > 0$, we have $\gcd(x+1, x) = 1$; this is because $\gcd(x+1, x) = \gcd(x, 1)$. In particular, we have $\gcd(k, k+1) = \gcd(k+1, k+2) = 1$.

For contradiction, assume $\gcd(k(k+2), k+1) \neq 1$. Then there exists a prime p such that $p \mid k(k+2)$ and $p \mid k+1$. Notice that since p is prime, $p \mid k(k+2)$ implies $p \mid k$ or $p \mid k+2$. However, since $\gcd(k, k+1) = 1$, we can't have $p \mid k$. Similarly, since $gcd(k+1, k+2) = 1$, we can't have $p \mid k+2$. Hence, $\gcd(k(k+2), k+1) = 1$, i.e. $k+1$ and k^2+2k are coprime.

(d) (2 points) Compute the following: $\left(\sum_{i=1}^{10} i^{16}\right) \mod 17$.

Answer: 10. Using FLT: $a^{p-1} \equiv 1 \pmod{p}$ for prime number p and $a \in \{1, 2, \ldots, p-1\}$, we get

$$\sum_{i=1}^{10} i^{16} \equiv \sum_{i=1}^{10} 1 = 10 \pmod{17}.$$

Therefore, $\left(\sum_{i=1}^{10} i^{16}\right) \mod 17 = 10$.

(e) (3 points) Solve the following system of congruences for x (i.e. solve for the unique solution modulo 60).

$$\begin{aligned} x &\equiv 2 \pmod{3}, \\ x &\equiv 3 \pmod{4}, \\ x &\equiv 4 \pmod{5}. \end{aligned}$$

Answer: $x \equiv 59 \pmod{60}$. Using the notations from CRT lecture, we have

$$M = 3 \cdot 4 \cdot 5 = 60$$

$$M_1 = \frac{M}{3} = 20,\ M_2 = \frac{M}{4} = 15,\ M_3 = \frac{M}{5} = 12.$$

Compute an inverse of 20 modulo $m_1 = 3$. Notice that $20 \cdot 2 = 40 \equiv 1 \pmod{3}$, so let $y_1 = 2$.

Similarly, we want to compute an inverse of 15 modulo $m_2 = 4$. Since $15 \cdot 3 = 45 \equiv 1 \pmod{4}$, let $y_2 = 3$.

Lastly, compute an inverse of 12 modulo $m_3 = 5$. Since $12 \times 3 = 36 \equiv 1 \pmod{5}$, let $y_3 = 3$.

Hence, $2y_1M_1 + 3y_2M_2 + 4y_3M_3 = 2 \cdot 2 \cdot 20 + 3 \cdot 3 \cdot 15 + 4 \cdot 3 \cdot 12 = 80 + 135 + 144 \equiv 20 + 15 + 24 = 59 \pmod{60}$.

Alternative solution: Let $x' = x + 1$. Consider the new system of congruences involving x':

$$x' \equiv 0 \pmod{3},$$

$$x' \equiv 0 \pmod{4},$$

$$x' \equiv 0 \pmod{5}.$$

Then $x' = 0$ is a solution. By CRT, $x' \equiv 0 \pmod{60}$, which gives $x \equiv -1 \equiv 59 \pmod{60}$.

(f) (3 points) Let p, q and r be polynomials of degree at most 2 over $GF(23)$ such that

$$\begin{aligned} p(1) = 1 \quad p(2) = 5 \quad p(3) = 4 \\ q(1) = 3 \quad q(2) = 7 \quad q(3) = 15 \\ r(1) = 1 \quad r(2) = 3 \quad r(3) = 1 \end{aligned}$$

Let $f(x) = p(x) + 2q(x) - 5r(x)$. Find $f(x)$. **Simplify your answer for full credit.**

Hint: You do not need to calculate the polynomials p, q, r.

Answer: $f(x) = 2x$. We can compute the values of f at $1, 2$ and 3 directly by evaluating:

$$\begin{aligned} f(1) &= 1 + 2 \cdot 3 - 5 \cdot 1 \mod 23 = 2 \\ f(2) &= 5 + 2 \cdot 7 - 5 \cdot 3 \mod 23 = 4 \\ f(3) &= 4 + 2 \cdot 15 - 1 \cdot 5 \mod 23 = 6 \end{aligned}$$

We know f has degree at most 2 because the degree of $p + q$ is at most $\max(p, q)$. Thus, there is only one polynomial which satisfies $f(1) = 2,\ f(2) = 4,\ f(3) = 6$ in $GF(23)$. Clearly, the polynomial $f(x) = 2x$ satisfies this.

(g) (2 points) Alice wants to send Bob a message of length n while guarding against k_e erasure errors and k_g general errors. How many total packets does she need to send?

Answer: $n + k_e + 2k_g$. If m is the number of packets, after k_e erasures, one has $m - k_e$ packets; to recover from $2k_g$ errors, $m - k_e = n + 2k_g$ packets are required. So $m = n + k_e + 2k_g$.

2 (15 Points) Short Answers II

Unless otherwise stated, you must show all your work in order to get full credit.

(a) (2 points) We roll a 6-sided fair die 4 times. How many possible strictly ascending sequences of numbers are there?

Answer: $\binom{6}{4} = \frac{6!}{2!4!}$. We can choose 4 out of the 6 numbers $\{1,2,3,4,5,6\}$. Then there is only one ascending order that we can have, hence the answer is $\binom{6}{4} = \frac{6!}{2!4!}$.

(b) (3 points) We choose n numbers from $\{1,2,\ldots,9\}$ uniformly at random with replacement and define x as the product of these numbers. What is the probability that x is divisible by 35? (n is a positive integer.)

Answer: $1-2\left(\frac{8}{9}\right)^n+\left(\frac{7}{9}\right)^n$. We will compute the probability of the complement which is the event that we do not choose both 5 or 7 in our process, which is equivalent to not choosing 5, or not choosing 7.

$$\mathbb{P}[x \text{ is divisible by } 35] = 1 - \mathbb{P}[\text{no } 5 \cup \text{no } 7]$$

By inclusion/exclusion, $\mathbb{P}[\text{no } 5 \cup \text{no } 7]$ is the probability of not choosing 5, $\left(\frac{8}{9}\right)^n$, plus the probability of not choosing 9, also $\left(\frac{8}{9}\right)^n$, minus the probability of both, $\left(\frac{7}{9}\right)^n$.

$$\mathbb{P}[\text{no } 5 \cup \text{no } 7] = \mathbb{P}[\text{no } 5] + \mathbb{P}[\text{no } 7] - \mathbb{P}[\text{no } 5 \cap \text{no } 7] = \left(\frac{8}{9}\right)^n + \left(\frac{8}{9}\right)^n - \left(\frac{7}{9}\right)^n,$$

where this is the complement probability. So the answer is

$$\mathbb{P}[x \text{ is divisible by } 35] = 1 - 2\left(\frac{8}{9}\right)^n + \left(\frac{7}{9}\right)^n.$$

(c) (3 points) Give a combinatorial proof for the following equation. (An algebraic proof receives 0 points.)

$$\binom{n}{2} = \sum_{i=1}^{n-1} i$$

Answer: LHS: Choose a pair from n items.
RHS: The first item can be paired with $n-1$ items, the second items can be paired with $n-2$ items to make a new pair, ..., the i^{th} item can be paired with $n-i$ items to create new pairs and so one. This is the $\sum_{i=1}^{n-1}(n-i) = \sum_{i=1}^{n-1} i$.

Alternatively: RHS: We divide the pairs according to the lowest number that is in a pair. The number of pairs where the lowest numbered item is i, is $n-i$ as there are $n-i$ items that have a higher number. This is the $\sum_{i=1}^{n-1}(n-i) = \sum_{i=1}^{n-i}$.

(d) (3 points) How many subsets of $\{1,2,...,2n\}$ are there that do not contain any elements x and y satisfying the equation $x+y=2n+1$? For example, $x=1$ and $y=2n$ cannot be in the subset at the same time. (n is a positive integer.)

Answer: $\sum_i \binom{n}{i} 2^i = 3^n$. We must ensure that x and $y = 2n+1-x$ are not both chosen for $x = 1,2,\ldots,n$. We may either choose x or choose y or choose neither. Hence, we have 3 options for each pair of (x,y) and there are n such pairs. Hence we can make 3^n subset with this property.

Alternatively: Assume we have n pairs $\{(1,2n),(2,2n-1),\ldots,(n,n+1)\}$. Since we can have at most one from each pair in our desired subset we can count the number of subsets by choosing i pairs out of the given n pairs, $\binom{n}{i}$, and discard the remaining pairs. Then we have two options to choose from each pair so $\binom{n}{i}2^i$. Since the number of pairs we choose can be any number from $i = 0$ to $i = n$ then by the binomial theorem we get $\sum_{i=0}^{i=n} \binom{n}{i} 2^i = 3^n$.

(e) (2 points) We have n fair coins and m biased coins in a bag. The biased coins land on heads with probability p. Without looking, we pick one uniformly at random and flip it. It lands heads. What is the probability that it is a fair coin, given that it landed heads? Leave your answer in terms of n, m and p.

Answer: $\dfrac{n}{n+2pm}$. Let F denote the event that we picked a fair coin and H denote the event that we flipped a heads.

$$\mathbb{P}[F|H] = \frac{\mathbb{P}[H|F]\mathbb{P}[F]}{P[H]} = \frac{(\frac{1}{2})(\frac{n}{n+m})}{(\frac{1}{2})(\frac{n}{n+m}) + p \cdot \frac{m}{n+m}} = \frac{n}{n+2pm}$$

(f) (2 points) Let $X \sim \text{Poisson}\left(\frac{1}{2}\right)$. Calculate $\mathbb{E}[X!]$. (Note: we really mean $X!$, as in "the factorial of X".) Recall that for $|r| < 1, \sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$

Answer: $2e^{-\frac{1}{2}}$. For $X = x$ we have $\mathbb{P}[X = x] = \dfrac{(\frac{1}{2})^x}{x!} e^{-\frac{1}{2}}$, so

$$\mathbb{E}[X!] = \sum_{x=0}^{\infty} x!\mathbb{P}[X = x] = \sum_{x=0}^{\infty} x! \frac{(\frac{1}{2})^x}{x!} e^{-\frac{1}{2}} = \frac{e^{-\frac{1}{2}}}{1-\frac{1}{2}} = 2e^{-\frac{1}{2}}$$

3 (6 Points) Joints

Let X and Y be two continuous random variables. The joint density of (X, Y) is uniform on the shaded region below, and 0 outside the shaded region. Mathematically, the figure consists of a rectangle.

(a) (1 point) What is the joint density $f_{X,Y}$ on the shaded region?

Answer: $\frac{1}{2}$, because the density must integrate to 1.

(b) (2 points) Set up, but do not evaluate the integrals for the values of $f_X(x)$ and $f_Y(y)$ on the shaded region.

Answer:

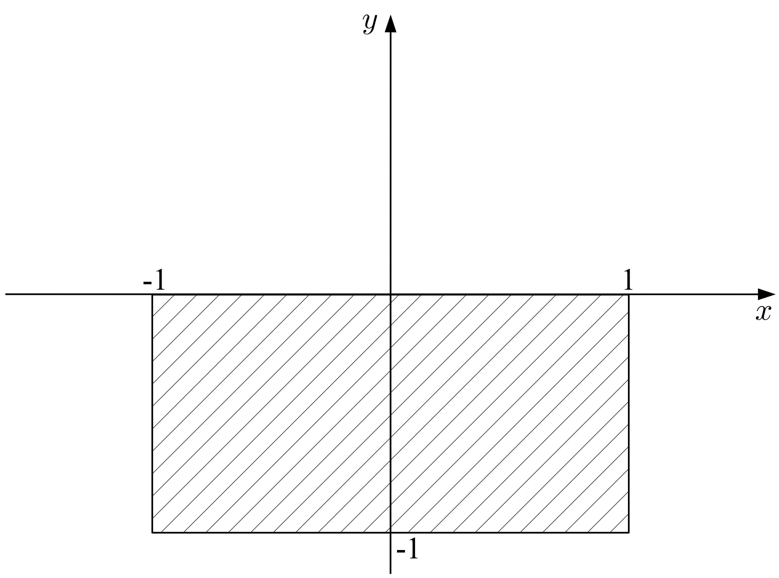
$$f_X(x) = \int_{-1}^{0} \frac{1}{2} dy$$
$$f_Y(y) = \int_{-1}^{1} \frac{1}{2} dx$$

(c) (3 points) Are X and Y independent? **Justify your answer.**

Answer:

$$f_X(x) = \int_{-1}^{0} \frac{1}{2} dy = \frac{1}{2}$$
$$f_Y(y) = \int_{-1}^{1} \frac{1}{2} dx = \frac{1}{1}$$

Since $f_{X,Y} = f_X(x) f_Y(y)$, X and Y are independent. Alternatively, one may compute the conditional densities and show that they are the same as the marginals on the shaded region.



4 (5 Points) Cubes

Suppose you have a $3 \times 3 \times 3$ inch cube of solid brown wood. You paint all 6 faces white and chop it up into $1 \times 1 \times 1$ inch subcubes. There are 27 subcubes in total. You then toss all of these in a bag, and with your eyes closed, you take one out and roll it. Opening your eyes, you notice that the 5 faces that are showing are brown. What's the probability that the face you can't see (i.e. 6^{th} face) is also brown? **Show all your work for full credit.**

Answer: Define the following events.

- Let O be the event that you observe 5 brown faces.
- Let B_5 be the event that the subcube you drew has 5 brown faces. There are 6 such cubes, so $P(B_5) = \frac{6}{27}$.
- Let B_6 be the event that the subcube you drew has 6 brown faces. $P(B_6) = \frac{1}{27}$.

$$P(B_6 \mid O) = \frac{P(O \mid B_6)P(B_6)}{P(O \mid B_6)P(B_6) + P(O \mid B_5)P(B_5)}$$

$P(O \mid B_6) = 1$ and $P(O \mid B_5) = \frac{1}{6}$. The answer is

$$\frac{1 \cdot \frac{1}{27}}{1 \cdot \frac{1}{27} + \frac{1}{6} \cdot \frac{6}{27}} = \frac{1}{2}$$

5 (12 Points) If It's Any Indication

Let G be an undirected graph on n vertices where each of possible edge of the graph is included with probability p, independent from every other edge. This means that for each edge e of the $\binom{n}{2}$ total edges, that edge e is a part of G with probability p, and is missing from the graph with probability $1-p$. A vertex is called *isolated* if it is adjacent to no vertices of the graph.

For the following parts, your answer should be an expression in terms of n and p. Show all your work for full credit.

(a) (2 points) Find the expected degree of a vertex.

Answer: Let D be the degree of the vertex v. Note that each of the $n-1$ other vertices are adjacent to v independently with probability p. This means that $D \sim \text{Binomial}(n-1, p)$, so $\mathbb{E}[D] = (n-1)p$.

Alternatively: Let

$$D = I_1 + \ldots + I_{n-1},$$

where I_j is the indicator that the jth other vertex is adjacent to v. Then

$$\begin{aligned}\mathbb{E}[D] &= \mathbb{E}[I_1 + \ldots + I_{n-1}] \\ &= \mathbb{E}[I_1] + \ldots + \mathbb{E}[I_{n-1}] \\ &= (n-1)\mathbb{E}[I_1] \\ &= (n-1)p.\end{aligned}$$

(b) (2 points) Find the variance in the degree of a vertex.

Answer: Using the fact that $D \sim \text{Binomial}(n-1, p)$, $\text{Var}[D] = (n-1)p(1-p)$.

Alternatively: Use the definition from part (a) and the independence of the indicators. Recall that the variance of a Bernoulli(p) random variable is $p(1-p)$.

$$\begin{aligned}\text{Var}[D] &= \text{Var}\left[I_1 + \ldots + I_{n-1}\right] \\ &= \text{Var}[I_1] + \ldots + \text{Var}[I_{n-1}] \\ &= (n-1)\text{Var}[I_1] \\ &= (n-1)p(1-p).\end{aligned}$$

(c) (4 points) Find the expected number of isolated vertices.

Answer: Let M be the number of isolated vertices in the graph. Then

$$M = I_1 + \ldots + I_n,$$

where I_j is the indicator of the event that the jth vertex is isolated. A vertex is isolated if each of the $n-1$ neighbors are not adjacent to it, each with probability $1-p$. Therefore

$$I_j = \begin{cases} 1 & \text{with probability } (1-p)^{n-1} \\ 0 & \text{with probability } 1-(1-p)^{n-1}. \end{cases}$$

By linearity of expectation,

$$\begin{aligned}\mathbb{E}[M] &= \mathbb{E}[I_1 + \ldots + I_n] \\ &= \mathbb{E}[I_1] + \ldots + \mathbb{E}[I_n] \\ &= n \cdot \mathbb{E}[I_1] \\ &= n(1-p)^{n-1}.\end{aligned}$$

(d) (4 points) Find the variance of the number of isolated vertices.

Answer: Note that the indicators are dependent; if we know that $n-1$ of the vertices are isolated, then the last vertex must also be isolated. In general, a vertex being isolated will increase the probability that other vertices are isolated.

Use the definition from part (c) and account for the fact that indicators are dependent.

$$\text{Var}[M] = \mathbb{E}[M^2] - \mathbb{E}[M]^2$$

We have $\mathbb{E}[M]$ from part (b), and we can calculate $\mathbb{E}[M^2]$ by multiplying out the product of indicators.

$$\begin{aligned}\mathbb{E}[M^2] &= \mathbb{E}[(I_1 + \ldots + I_n)^2] \\ &= \mathbb{E}\left[\sum_{i=1}^{n} I_i^2\right] + \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j\neq i}^{n} I_i I_j\right] \\ &= \sum_{i=1}^{n} E[I_i^2] + \sum_{i=1}^{n}\sum_{j\neq i}^{n} \mathbb{E}[I_i I_j] \\ &= n \cdot \mathbb{E}[I_1^2] + n(n-1)\mathbb{E}[I_1 I_2] \\ &= n \cdot \left((1-p)^{n-1}\right) + n(n-1) \cdot \left((1-p)^{n-1}(1-p)^{n-2}\right) \\ &= n(1-p)^{n-1} + n(n-1)(1-p)^{2n-3}.\end{aligned}$$

$$\text{Var}[M] = n(1-p)^{n-1} + n(n-1)(1-p)^{2n-3} - (n(1-p)^{n-1})^2$$

Alternatively: Break the variance into a sum of covariances.

$$\begin{aligned}\text{Var}[M] &= \text{Var}[I_1 + \ldots + I_n] \\ &= \sum_{i=1}^{n} \text{Var}[I_i] + \sum_{i=1}^{n}\sum_{j\neq 1}^{n} \text{Cov}[I_i, I_j] \\ &= n \cdot \text{Var}[I_1] + n(n-1) \cdot \text{Cov}[I_1, I_2].\end{aligned}$$

$$\text{Var}[I_1] = (1-p)^{n-1}\left(1-(1-p)^{n-1}\right).$$

$$\begin{aligned}\text{Cov}[I_1, I_2] &= \mathbb{E}[I_1 I_2] - \mathbb{E}[I_1]\mathbb{E}[I_2] \\ &= (1-p)^{n-1}(1-p)^{n-2} - \left((1-p)^{n-1}\right)^2 \\ &= (1-p)^{2n-3} - (1-p)^{2n-2} \\ &= (1-p)^{2n-3}\left(1-(1-p)^1\right) \\ &= p(1-p)^{2n-3}.\end{aligned}$$

$$\text{Var}[M] = n(1-p)^{n-1}\left(1-(1-p)^{n-1}\right) + n(n-1)p(1-p)^{2n-3}.$$

These answers are equivalent.

6 (10 Points) Markov Chains

Kevin and Shahzar are playing a game with 2 bins. On each turn, Shahzar picks a bin uniformly at random. If the bin already has a ball in it, Shahzar does nothing. If the bin doesn't have a ball, Shahzar throws a ball into that bin**. On the same turn, Kevin picks a bin uniformly at random, independently of Shahzar, and empties it (removes any balls that are in the bin). For each turn, Shahzar goes first, then Kevin.

** Note that this means the maximum number of balls in any bin is 1.

(a) (4 points) For a particular bin, construct a two-state Markov chain for the number of balls it contains at the end of a turn. **Clearly indicate the states and transition probabilities of the chain.**

Answer: We can model a given bin as a two state Markov chain with states 0 and 1.

We first consider the probability that the bin is empty gets a ball thrown into it. This happens only when Shahzar picks the bin and Kevin doesn't.

$$\mathbb{P}(0 \to 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

So $\mathbb{P}(0 \to 1) = 1/4$ and $\mathbb{P}(0 \to 0) = 3/4$.

Now we consider the probability that the bin has a ball in it and becomes empty. This happens if Kevin picks the bin.

$$\mathbb{P}(1 \to 0) = \frac{1}{2}$$

So $\mathbb{P}(1 \to 0) = 1/2$ and $\mathbb{P}(1 \to 1) = 1/2$.

We can put this information into a probability transition matrix:

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

While this is the correct answer, we did not carry forward any penalty for parts (b) and (c), so long as your answer to part (a) was a valid Markov chain on two states. In other words, for parts (b) and (c), grading was based on your answer for part (a) assuming a valid chain, and not the correct chain.

(b) (2 points) Does the Markov chain converge to a unique invariant distribution? **Justify your answer.**

Answer: Yes, the Markov chain has an invariant distribution because it is irreducible and aperiodic. We can see that it is irreducible because each state communicates to the other state.

To be explicit, we construct the loop $0 \to 1 \to 0$ through the Markov Chain, which visits every state once and loops back on itself. This means each state can communicate with each other state.

We can see that it is aperiodic because each state has a path to itself (self-loop). To be explicit, note that $0 \to 0$ is a valid path to return to state 0. So $d(0) = \gcd\{1, 2, \dots\} = 1$. Therefore the chain is aperiodic.

(c) (4 points) Find the invariant distribution(s) of the Markov chain. **Show all your work for full credit.**

Answer: We solve for the invariant distribution π of the Markov chain.

$$\pi = [\pi_0 \quad \pi_1] = [\pi_0 \quad \pi_1] \cdot \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$$

So we have the following equations:

$$\pi_0 = \frac{3\pi_0}{4} + \frac{\pi_1}{2}$$
$$\pi_1 = \frac{\pi_0}{4} + \frac{\pi_1}{2}$$

Solving, we get $\pi_0 = 2\pi_1$. Since $\pi_0 + \pi_1 = 1$, we have $2\pi_0 + \pi_1 = 1$, so $\pi_1 = 1/3$ and $\pi_0 = 2/3$.

7 (11 points) Distributed Distribution

Robel took his Organic Chemistry exam and wanted to see the distribution of scores. Since the web portal site was down, Robel was unable to access the true distribution. Robel decided to survey students, and get an estimate of the distribution for himself.

Let n be the number of students who fill out his survey. For each student i who fills out his survey, the score they report X_i follows $X_i \sim \text{Exp}\left(\frac{1}{\mu}\right)$, where μ is the true mean score on the exam. All X_i are independent and identically distributed (i.i.d.), with mean μ and variance μ^2. Note that this means the reported scores can be anywhere in $[0, \infty)$.

For the following parts, show all your work for full credit.

(a) (2 points) Robel was looking through the survey results as they came in and saw a score above 90. However, he does not remember the exact score. What is the expected value of the reported score he saw, given it was above 90? You may use μ in your answer.

Answer: Let T be the random variable corresponding to the score he saw. We know that $T \sim \text{Exp}\left(\frac{1}{\mu}\right)$. We are interested in $\mathbb{E}[T \mid T > 90]$. But this is the same as $90 + \mathbb{E}[T']$, where T' is a fresh exponential, since the exponential distribution is memoryless. Thus, the answer is $90 + \mu$.

Robel is now interested in estimating the true mean of the distribution. It then makes sense to consider $\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, which can be interpreted as the sample mean of the reported scores from the survey.

(b) (2 points) **Determine $\mathbb{E}[\hat{\mu}_n]$, and $\text{Var}(\hat{\mu}_n)$.**

Answer:

$$\mathbb{E}[\hat{\mu}_n] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \frac{1}{n}\cdot n \cdot \mu = \mu$$

$$\text{Var}(\hat{\mu}_n) = \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(X_i) = \frac{1}{n^2}\cdot n \cdot \mu^2 = \frac{\mu^2}{n}$$

The first line is by linearity of expectation. The second line is due to the fact that the variance of a sum of independent random variables is the sum of the variances.

(c) (3 points) Using Chebyshev's inequality, find the tightest upper bound on the probability that the sample mean $\hat{\mu}_n$ is at least 10 points above the true mean μ.
Answer:

$$P(\hat{\mu}_n - \mu \geq 10) \leq P(|\hat{\mu}_n - \mu| \geq 10) \leq \frac{\text{Var}(\hat{\mu}_n)}{10^2} = \frac{\frac{\mu^2}{n}}{100} = \frac{\frac{\mu^2}{n}}{100} = \frac{\mu^2}{100n}$$

Since there is no guarantee of symmetry, the best bound we can get with Chebyshev's is the two-tail bound $P(|\hat{\mu}_n - \mu| \geq 10) \leq \frac{\mu^2}{100n}$. Note that if the distribution were symmetric about μ, then we would be able to come up with the tighter one-tail bound $P(\hat{\mu}_n - \mu \geq 10) \leq \frac{\mu^2}{2\times 100n}$, but we CANNOT do that since the exponential distribution is not symmetric about its mean.

(d) (4 points) Robel knows that 100 students filled out his survey. He later finds out from the web portal site that the true mean was actually 50. Using the Central Limit Theorem as a means of approximation, **What the probability that $\hat{\mu}_n$ is *not* within 1 point of the true mean $\mu = 50$? Leave your answer in terms of the standard normal CDF Φ.**

Answer:

$$P(|\hat{\mu}_n - \mu| \geq 1) = 2P\left(\frac{\hat{\mu}_n - \mu}{\mu/\sqrt{n}} \geq \frac{1}{\mu/\sqrt{n}}\right)$$

Let $Z = \dfrac{\hat{\mu}_n - \mathbb{E}[\hat{\mu}_n]}{\sqrt{\mathrm{Var}(\hat{\mu}_n)}} = \dfrac{\hat{\mu}_n - \mu}{\mu/\sqrt{n}}$.
The CLT states that $Z \sim \mathcal{N}(0,1)$.

$$\begin{aligned}
P(|\hat{\mu}_n - \mu| \geq 1) &\approx 2P\left(Z \geq \frac{\sqrt{n}}{50}\right) \\
P\left(Z \geq \frac{\sqrt{n}}{50}\right) &= P\left(Z \geq \frac{\sqrt{100}}{50}\right) \\
&= P(Z \geq 0.2) \\
&= 1 - \Phi(0.2) \\
\Rightarrow P(|\hat{\mu}_n - \mu| \geq 1) &\approx 2(1 - \Phi(0.2))
\end{aligned}$$

8 (11 points) Graphs, Polynomials, and Counting: The Ultimate CS70 Crossover!

Recall that a graph is n-colorable if you can color the vertices using n colors such that no adjacent vertices have the same color. Now instead of just validity, we can count also *how many* colorings (if any) exist. We define $P_G(x)$ as the number of ways of coloring a graph G with x colors. For example, if $P_G(n) = 0$ for a graph G and a positive integer n, then G is not n-colorable (there are zero ways to n-color G).

It turns out that this function is always a polynomial for a given graph G. Assume vertices are distinguishable. **For each of the following parts, show all your work for full credit.**

(a) (2 points) Let G be the triangle graph (a complete graph of 3 vertices). Calculate $P_G(5)$.

Answer: 60. There are 5 choices of color for the first vertex, then 4 choices for the next vertex, then 3 for the last vertex. Thus, by first rule of counting, $5 \cdot 4 \cdot 3 = 60$ colorings.

(b) (4 points) Let G be a complete graph of n vertices.

(i) Find the polynomial $P_G(x)$.
Answer: There are x choices for the first vertex, then $x-1$ for the next vertex and so on. Thus $P_G(x)$ is $x \cdot (x-1) \cdot (x-2) \cdot \ldots \cdot (x-(n-1))$.

(ii) Use the polynomial you found in part (i) to show that the minimum number of colors needed to color the complete graph of n vertices is n.

Answer: Note that if $P_G(x) \geq 0$ for some x, that means the graph can be colored with x colors. In this case, since $0, 1, 2, \ldots, n-1$ are all roots of the polynomial, they each have 0 colorings. n is the smallest value of x such that $P_G(x)$ is positive, and thus the graph is n-colorable.

(c) (3 points) Let G be a tree of n vertices. Find the polynomial $P_G(x)$.

Answer: $x(x-1)^{n-1}$. Pick a root, and we will first color the root, and then color vertices in depth 1, and then color vertices in depth 2, and so on.

There are x choices for the root. When we color a vertex v in depth k, notice that v is adjacent to exactly one colored vertex (which is a vertex in depth $k-1$), hence there are $x-1$ choices for v. This is true for every non-root vertex in the graph. Thus $P_G(x) = x(x-1)^{n-1}$. (This is like HW4 Q2b!).

(d) (2 points) Let G be a graph of n vertices that contains at least one edge. Prove that the sum of the coefficients in $P_G(x)$ is 0.

Answer: $P_G(1) = a_n + a_{n-1} + \ldots + a_0$ is the number of ways to color G with 1 color. If G is non-empty, it contains an edge, then it can not be colored with 1 color so this sum must be 0.

9 (11 points) Lineage Tracing

Consider the following model for tracking cell division. A petri dish begins with a single white cell. At the start of each time step $t \geq 1$, every cell in the petri dish will divide into two cells that inherit its color**. Then, each white cell in the petri dish will become green with probability p, independently. Then the timestep ends.

**This means at the end of any timestep t, there are 2^t cells in the dish.

For each of the following parts, show all your work for full credit.

(a) (2 points) What is the probability that a particular cell at the end of timestep n is green?

Answer: The probability that a particular cell at the end of timestep n is green is $\mathbb{P}[geom(p) \leq n] = 1-(1-p)^n$.

(b) (3 points) The **descendants** of a cell are the two cells that it divided into and all of their descendants. A cell is an **ancestor** of any of its descendants. Note that the first cell in the petri dish is the ancestor of all cells.

Suppose we observed that a cell is green at the end of time step n. Given this observation, what is the probability that the green mutation occurred in an ancestor of the observed cell at timestep t, where $t \leq n$?

Answer: If $X \sim \text{Geometric}(p)$, the probability that the mutation occurred in an ancestor at level t is equaled to $\mathbb{P}[X = t \mid X \leq n]$ which is:

$$\frac{\mathbb{P}[X = t \cap X \leq n]}{\mathbb{P}[X \leq n]} = \frac{p(1-p)^{t-1}}{1-(1-p)^n}$$

(c) (3 points) What is the expected number of green cells at the end of time step n? Give your answer as an expression in terms of n and p.

Answer: Let Y_i be the indicator for the i^{th} cell at the end of timestep n. Then the desired quantity is $\mathbb{E}[\sum_{i=1}^n Y_i] = \sum_{i=1}^n \mathbb{E}[Y_i]$. The probability that a particular cell at the end of timestep n is green is $\mathbb{P}[geom(p) \leq n] = 1-(1-p)^n$. So, $\mathbb{E}[Y_i] = 1-(1-p)^n$. The number of cells at the end of timestep n is 2^n. Thus, we have that the expected number of green cells at the end of timestep n is $2^n(1-(1-p)^n)$

(d) (3 points) Suppose $p > 1/2$. Prove that as n tends to ∞, the probability that every cell at time step n is green tends to 1.

Answer: Let W be the number of white cells at level n. We must show that as $n \to \infty$, $Pr[W \geq 1] \to 0$. We can do this two ways:

Using indicators like last time, we see that $E(W) = (2(1-p))^n$ which tends to 0 as n tends to ∞ since $p > 1/2$ implies that $2(1-p) < 1$. Using Markov's inequality, we see that:

$$\mathbb{P}[W \geq 1] \leq \frac{\mathbb{E}(W)}{1} = (2(1-p))^n \to 0$$

Alternatively, we can note that for each cell, the probability of not being green is $(1-p)^n$ so union bounding over all 2^n cells, we have that the probability that any cell remains not green is at most $(2(1-p))^n$ which tends to 0 as $n \to \infty$

Submission

- Keep the recording going;
- Scan answer booklet and cheatsheets into PDF;
- Submit to Gradescope by 11:15PM PDT. If Gradescope is being really slow, you may submit your exam PDF using a private Piazza post and attaching your pdf to the private post;
- Stop the recording;
- Upload recording to your Google Drive, Box, DropBox, etc;
- Submit the link to your uploaded recording using this form:
  by 11:59PM 8/14 PDT latest;
- Submit your cheatsheets to Gradescope under "Final Cheat Sheets". If you did not use any cheatsheets, you must submit 6 blank pages. If you already submitted the cheatsheets and did not make any changes, then you do not need to resubmit.

If you have technical issues during the exam, you should report these issues when you submit your exam by making a private post on Piazza. You should attach your exam pdf to this post if you have it.

You are strong!

Contributors:

- Kevin Zhu.
- Amin Ghafari.
- Sagnik Bhattacharya.
- Khalil Sarwari.
- Yining Liu.
- Alan He.
- Shahzar Rizvi.
- Robert Wang.