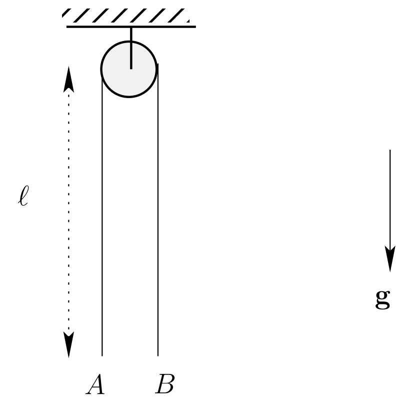
University of California, Berkeley
Mechanical Engineering
ME104 Engineering Mechanics **NAME** ______________________________
Test 1 S20 *Prof S. Morris*

1. (100) The chain of length 2ℓ and mass per unit length ρ hangs over the frictionless pulley in the equilibrium shown. If end A is given a small upward displacement, the imbalance causes the chain to accelerate, so that end A rises above end B. The mass and the diameter of the pulley are both negligibly small.



(a) Assuming that end A has risen through a distance x from its initial position, draw separate free body diagrams for the portions of the chain on each side of the pulley.

(b) Hence derive the differential equation giving the acceleration $\ddot{x}$ of the chain in terms of g, ℓ and the upward displacement x of end A.

(c) Find the speed of end A when it reaches the pulley.

<u>Solution</u>

(a) For both portions the tension T acting at the top acts upwards; because the pulley is massless and frictionless, this tension is same for both portions. The other force acting is the weight; for left portion of chain, this is $\rho g(\ell - x)g$; for the right, $\rho g(\ell + x)$.

(b) From the free body diagram for the left portion of chain

$$\rho(\ell - x)\ddot{x} = T - \rho g(\ell - x); \tag{1.1a}$$

from that for the right hand portion

$$\rho(\ell + x)(-\ddot{x}) = T - \rho g(\ell + x) \tag{1.1b}$$

The acceleration is $+\ddot{x}$ for the left side, but $-\ddot{x}$ for the right.

The equation giving $\ddot{x}$ in terms of x, g and ℓ is obtained by eliminating T between (1.1b) and (1.1a). It is

$$\ddot{x} = \frac{g}{\ell}x \tag{1.2}$$

(c) Because (1.2) determines the acceleration as a function of distance x, use the chain rule in the form

$$\ddot{x} = \dot{x}\,\frac{\mathrm{d}}{\mathrm{dx}}\dot{x}, = \frac{\mathrm{d}}{\mathrm{dx}}\Big(\frac{1}{2}\dot{x}^2\Big). \tag{1.3}$$

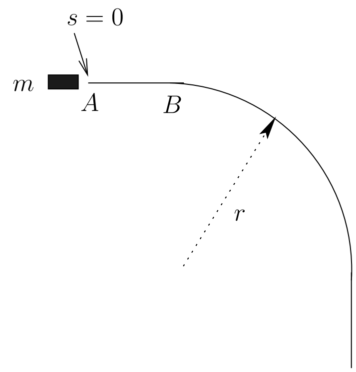
By substituting (1.3) into (1.2), then integrating and applying the initial condition $\dot{x} = 0$ at $x = 0$,

$$\frac{1}{2}\dot{x}^2 = \frac{g}{2\ell}x^2 \ \Rightarrow \dot{x} = x\sqrt{\frac{g}{\ell}} \tag{1.4a, b}$$

Equation (1.4b) gives the speed $\dot{x}$ as a function of the displacement x of end A.

When A reaches the top, $x = \ell$ and $\dot{x} = \sqrt{g\ell}$.

2. (100) The car of mass m travelling at speed v_A on a horizontal road enters the bend having the form of a quarter circle. To stop the vehicle, the driver begins to apply the brakes at point A before entering the bend at B. The brakes are applied so that the vehicle does not skid, and the resultant friction force exerted by the road on the vehicle is constant in magnitude until the vehicle stops. The coefficient of friction is μ (dimensionless). (Gravity acts into the page, so that the car is travelling in the horizontal plane.)



(a) Find, as a function of v_A, μ, g and the distance s_B from A to B, the speed v_B at which the vehicle enters the bend.

(b) Now assuming the vehicle to have entered the bend, draw the free–body diagram showing the normal and tangential components of the single force (friction) acting on the vehicle in the horizontal plane.

(c) How would the vehicle respond if $v_B^2 > \mu g r$? Explain your answer physically.

(d) Using the free–body diagram drawn in (b), derive the differential equation determining the vehicle speed v as a function of distance s and the parameters g, μ and r..

(e) By solving the equation obtained in part (d), derive the formula giving the stopping distance s in terms of g, μ, r, s_B and v_B.

Given:

$$\int \frac{\mathrm{d}z}{(1-z^2)^{1/2}} = \sin^{-1} z + \text{const.}$$

(Dummy variable z; the function denoted by $\sin^{-1} z$ in your text is also denoted by $\arcsin z$ by some authors.)

<u>*Solution*</u>

(a) Because the section AB of road is straight and the vehicle is decelerating, the force $\mu m g$ exerted by friction between the road and the tyres is antiparallel to the acceleration. The equation of motion is therefore

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\mu g. \tag{2.1}$$

To obtain the equation determining v as a function of distance s use the chain rule in the form

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}s}, = v\frac{\mathrm{d}v}{\mathrm{d}s}, = \frac{\mathrm{d}}{\mathrm{d}s}\big(\frac{1}{2}v^2\big). \tag{2.2a, b, c}$$

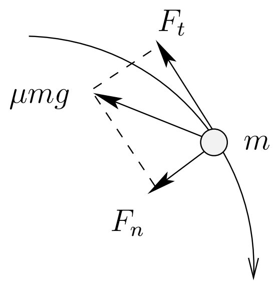
Then

$$\frac{\mathrm{d}(v^2)}{\mathrm{d}s} = -2\mu g,$$

so that $v_B^2 - v_A^2 = 2\mu g(s_A - s_B)$:

$$v_B = \sqrt{v_A^2 - 2\mu g(s_B - s_A)} \tag{2.3}$$

(b) The magnitude of the frictional force is μmg. It has two components: the normal component F_n is directed towards the centre of curvature of the bend; because the vehicle braking, the tangential component F_s is directed in the sense opposite to that of the velocity.



(c) If the car follows the bend at point B, its centripetal acceleration there would be v_B^2/r. If $v_B^2 > \mu gr$, this acceleration would be too large to be balanced by the maximum frictional force, and the vehicle would skid.

(d) The normal n and the tangential s components of Newton's second law are:

$$m\frac{v_2}{r} = F_n \tag{2.4a}$$

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -|F_s| \tag{2.4b}$$

In (2.4b), the minus sign is necessary because F_s here is causing the vehicle to decelerate.

Because $F_n^2 + F_s^2 = (\mu mg)^2$, it follows from (2.4) that

$$\frac{v^4}{r^2} + \left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 = \mu^2 g^2 \tag{2.5}$$

By solving for $\frac{\mathrm{d}v}{\mathrm{d}t}$, and choosing the sign of the square root to ensure that $\frac{\mathrm{d}v}{\mathrm{d}t} < 0$,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\left[\mu^2 g^2 - \frac{v^4}{r^2}\right]^{1/2} \tag{2.6}$$

This equation determines the vehicle speed v as a function of time t, and the parameters g, μ and r.

By using the chain rule, as in equation (2.2),

$$\frac{\mathrm{d}(v^2)}{\mathrm{d}s} = -2\left[\mu^2 g^2 - \frac{v^4}{r^2}\right]^{1/2}, = -2\mu g\left[1 - \left(\frac{v^2}{\mu gr}\right)^2\right]^{1/2}. \tag{2.7a, b}$$

This equation determines v as a function of s, g, μ and r.

(e) Equation (2.7) is separable: it can be expressed in the form

$$\left[1 - \left(\frac{v^2}{\mu gr}\right)^2\right]^{-1/2} \mathrm{d}(v^2) = -2\mu g \mathrm{d}s, \tag{2.8}$$

so that the left side depends on v^2 only, and the right side on s only.

Let

$$z = \frac{v^2}{\mu gr}. \tag{2.9}$$

Then (2.9) is

$$\frac{\mathrm{d}z}{[1 - z^2]^{1/2}} = -\frac{2}{r}\mathrm{d}s.$$

By integrating from s_B (where $z = v_B^2/(\mu g r)$) to the stopping point s (where $v^2 = 0$),

$$\frac{r}{2}\int_{v_B^2/(\mu g r)}^{0} \frac{\mathrm{d}z}{[1-z^2]^{1/2}} = s - s_B. \tag{2.10}$$

The stopping distance s is therefore given by

$$s = s_B + \frac{r}{2}\int_{0}^{v_B^2/(\mu g r)} \frac{\mathrm{d}z}{\left[1-z^2\right]^{1/2}}, = \sin^{-1}\left[\frac{v_B^2}{\mu g r}\right] \tag{2.11a, b}$$

end