1. Check if the following matrices are invertible and find the inverse when possible

(a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, det = 1, invertible

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 4 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Inverse $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & -2 & -3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. det = 0 first row = - second row

not invertible

2. Find bases in Nul A and Col A for the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & 5 & 7 \\ -1 & 2 & 2 & 0 & 1 \end{bmatrix}.$$

echelon form $\begin{bmatrix} 1 & -2 & 3 & 5 & 7 \\ 0 & 0 & 5 & 5 & 8 \end{bmatrix}$

first and third columns give basis for Col A

$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$

To solve $A\bar{x} = \bar{0}$ free variables are x_2, x_4, x_5

$x_2 = 1,\ x_4 = x_5 = 0$ solution $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$x_2 = 0,\ x_4 = 1,\ x_5 = 0$ $\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

$x_2 = x_4 = 0,\ x_5 = 1$ $\begin{bmatrix} -2.2 \\ 0 \\ -1.6 \\ 0 \\ 1 \end{bmatrix}$

Basis of Nul A $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2.2 \\ 0 \\ -1.6 \\ 0 \\ 1 \end{bmatrix} \right\}$

3. Consider the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined by the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Write the matrix of T.

$T(\bar{e}_1) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$T(\bar{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(b) Describe the kernel and the range of T.

Kernel of $T = \{ (x_1, x_2) \mid 2x_1 + x_2 = 0 \}$

Span $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Range of $T =$ Span Col $A =$ Span$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

4. Let S be the circle in $\mathbb{R}^2$ with center at $(1, 1)$ and radius 3 and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find the area of $T(S)$.

Area $T(S) = \left|\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right| \cdot$ Area $S = 2 \times \pi \times 3^2 = 18\pi$

5. Let $\{v_1, v_2, v_3\}$ be a linearly independent set of vectors in a vector space V. Show that the set $\{v_1 + v_2, v_2 + v_3, v_1 + v_3\}$ is also linearly independent.

$x_1(\bar{v}_1 + \bar{v}_2) + x_2(\bar{v}_2 + \bar{v}_3) + x_3(\bar{v}_1 + \bar{v}_3) = \bar{0} \Rightarrow$

$(x_1 + x_3)\bar{v}_1 + (x_1 + x_2)\bar{v}_2 + (x_2 + x_3)\bar{v}_3 = \bar{0} \Rightarrow$

Since $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ are linearly independent

we have $x_1 + x_3 = x_1 + x_2 = x_2 + x_3 = 0 \Rightarrow x_3 = x_2,\ x_1 = x_3,$
$x_1 = -x_3$

$x_1 = x_2 = x_3 = 0.$