MIDTERM 2 SOLUTIONS

1. Let A be 8×10 matrix of rank 7. Determine which of the following statements are true and which are false and explain your answer.

(a) For every vector $\mathbf{b}$ in $\mathbb{R}^8$ the equation $A\mathbf{x}=\mathbf{b}$ has a solution.

(b) There are some vectors $\mathbf{b}$ in $\mathbb{R}^8$ such that $A\mathbf{x}=\mathbf{b}$ has no solutions.

(c) The equation $A\mathbf{x}=\mathbf{b}$ has exactly one solution for every $\mathbf{b}$ in $\mathbb{R}^8$.

(d) If the equation $A\mathbf{x}=\mathbf{b}$ is not inconsistent then it has infinitely many solutions.

Solution. (a) is false because $\operatorname{Col} A$ has dimension 7 and one can find $\mathbf{b}$ in $\mathbb{R}^8$ but not in $\operatorname{Col} A$. By the same reason (b) is true. (c) is false as we already explained that for some $\mathbf{b}$ the system is inconsistent. Finally, (d) is true. Indeed, $\operatorname{Nul} A$ has dimension 3. If $\mathbf{x}\in\mathbb{R}^{10}$ is a solution then $\mathbf{x}+\mathbf{y}$ is a solution for all $\mathbf{y}$ in $\operatorname{Nul} A$.

Another possible explanation: A has 7 pivot columns. There are 3 free variables and there is a zero row in the reduced echelon form of A.

2. (a) Find eigenvalues and eigenvectors of the matrix

$$A=\begin{bmatrix}0 & 3\\ 1 & 2\end{bmatrix}.$$

(b) Compute A^k for arbitrary positive k.

Solution.

(a) The characteristic equation

$$\det(A-\lambda I)=-\lambda(2-\lambda)-3=\lambda^2-2\lambda-3=0$$

has two roots $\lambda_1=3$ and $\lambda_2=-1$. The corresponding eigenvectors are $\mathbf{v}_1=\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and $\mathbf{v}_2=\left[\begin{smallmatrix}3\\-1\end{smallmatrix}\right]$.

(b) We have $A=PDP^{-1}$ where

$$D=\begin{bmatrix}3 & 0\\ 0 & -1\end{bmatrix},\quad P=\begin{bmatrix}1 & 3\\ 1 & -1\end{bmatrix}.$$

Therefore

$$A^k=PD^kP^{-1}=\begin{bmatrix}1 & 3\\ 1 & -1\end{bmatrix}\begin{bmatrix}3^k & 0\\ 0 & (-1)^k\end{bmatrix}\begin{bmatrix}\frac14 & \frac34\\ \frac14 & -\frac14\end{bmatrix}=\begin{bmatrix}\frac{3^k+3(-1)^k}{4} & \frac{3^{k+1}-3(-1)^k}{4}\\ \frac{3^k+(-1)^{k+1}}{4} & \frac{3^{k+1}+(-1)^k}{4}\end{bmatrix}.$$

3. Let L be the line in $\mathbb{R}^3$ spanned by the vector $\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3$ and $T(\mathbf{x})=\operatorname{proj}_L\mathbf{x}$ denote the orthogonal projection onto L.

(a) Compute the standard matrix A of the linear transformation T. (b) Find an orthogonal matrix P and a diagonal matrix D such $A=PDP^{-1}$.

Solution.(a) Use the formula $T(\mathbf{x}) = (\mathbf{x}\cdot(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3))(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)$. Get

$$T(\mathbf{e}_1) = T(\mathbf{e}_2) = T(\mathbf{e}_3) = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

Therefore

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

(b) We have to find an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ such that $\mathbf{v}_1 = c(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)$. We can choose $\mathbf{v}_2 = d(\mathbf{e}_1 - \mathbf{e}_2), \mathbf{v}_3 = f(\mathbf{e}_1+\mathbf{e}_2-2\mathbf{e}_3)$. Then choose constant c, d, f to get vectors of norm 1. Use $P = [\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3]$ to get

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

4. Find the equation $y = \beta_0 + \beta_1 x$ of the least squares line which best fits the data points

$$(1,0), (2,1), (4,2), (5,3).$$

Solution. We have to find the least squares solutions of the system

$$\beta_0 + \beta_1 = 0,\ \beta_0 + 2\beta_1 = 1,\ \beta_0 + 4\beta_1 = 2,\ \beta_0 + 5\beta_1 = 3.$$

Write it in the matrix form

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

Multiplying by the transposed matrix gives

$$\begin{bmatrix} 4 & 12 \\ 12 & 46 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 25 \end{bmatrix}$$

Solve by your favorite method to get $\beta_0 = -0.6, \beta_1 = 0.7$.

5. Let A be an $n\times n$ matrix satisfying the equation

$$(A - \lambda_1 I)(A - \lambda_2 I) = 0$$

for some real numbers λ_1, λ_2, such that $\lambda_1 \neq \lambda_2$. Show that A is diagonalizable. (Hint: show that $\mathrm{Nul}(A-\lambda_2 I)$ and $\mathrm{Col}(A - \lambda_2 I)$ are eigenspaces of A.)

Solution. Since $\mathrm{Nul}(A-\lambda_2 I)$ consists of vectors $\mathbf{v}$ such that $(A-\lambda_2 I)\mathbf{v} = \mathbf{0}$ which is equivalent to $A\mathbf{v} = \lambda_2\mathbf{v}$, we obtain that $\mathrm{Nul}(A - \lambda_2 I)$ consists of eigenvectors of A with eigenvalue λ_2. Assume that $\mathbf{w}$ lies in $\mathrm{Col}(A - \lambda_2 I)$ then $\mathbf{w} = (A - \lambda_2 I)\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n$. Then we get

$$(A - \lambda_1 I)\mathbf{w} = (A - \lambda_1 I)(A - \lambda_2 I)\mathbf{x} = 0.$$

Therefore $\mathbf{w}$ lies in $\mathrm{Nul}(A - \lambda_1 I)$ and hence $\mathbf{w}$ is an eigenvector of A with eigenvalue λ_1. This shows that $\mathrm{Col}(A - \lambda_2 I)$ is a subspace of the eigenspace with eigenvalue λ_1. If V_1 and V_2 are eigenspaces of A with eigenvalues λ_1 and λ_2 respectively, we get

$$\dim V_1 \geq \dim \mathrm{Nul}(A - \lambda_2 I), \quad \dim V_2 = \dim \mathrm{Col}(A - \lambda_2 I)$$

Hence

$$\dim V_1 + \dim V_2 \geq \dim \mathrm{Nul}(A - \lambda_2 I) + \dim \mathrm{Col}(A - \lambda_2 I) = n,$$

and therefore A is diagonalizable. Note that in fact we have the equality

$$\dim V_1 + \dim V_2 = \dim \mathrm{Nul}(A - \lambda_2 I) + \dim \mathrm{Col}(A - \lambda_2 I) = n.$$