EECS 126 Probability and Random Processes — UC Berkeley, Spring 2020
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Midterm 2

Last Name	First Name	SID

***Rules*.**

- **Unless otherwise stated, all your answers need to be justified.**
- You may reference your notes, the textbook, and any material that can be found through the course website.
- You may use Google to search up general knowledge. However, **searching up a question is not allowed**.
- **Collaboration with others is strictly prohibited**.
- You have exactly **100** minutes total to both write your exam **AND** submit the exam to Gradescope.
- For any clarifications you have, please create a private Piazza post. We will have a Google Doc that shows our official clarifications.

***Grading*.**

Problem	points earned	out of
Pledge		4
Problem 1		52
Problem 2		18
Problem 3		26
Total		100

1 Assorted Problems

(a) **Is this True?**

Consider some sequence of random variables $\{X_n\}$. Prove or provide a counterexample for the following.

(i) For this subpart ONLY, take a sequence of random variables $\{X_i\}$ with equal expectation, i.e. $\mathbb{E}[X_i] = a, \forall i$ for some constant a. If variance of X_n converges to 0, then X_n converges in probability to its expectation.

True. Prove by Chebyshev's (this is how WLLN is proved).

$$P(|X_n - \mathbb{E}[X_n]| > \epsilon) \leq \frac{\text{Var}(X_n)}{\epsilon^2} \to 0$$

(ii) For any sequence of $\{X_i\}$, convergence in probability to a constant, i.e. $P(|X_n - a| > \epsilon) \to 0$ as $n \to \infty$, implies the variance of X_n goes to 0 as $n \to \infty$.

False. Take a scaled Bernoulli, where X_n is defined as follows:

$$X_n = \begin{cases} n & \text{w.p. } \frac{1}{n^2} \\ 0 & \text{otherwise} \end{cases}$$

X_n converges to 0 in probability:

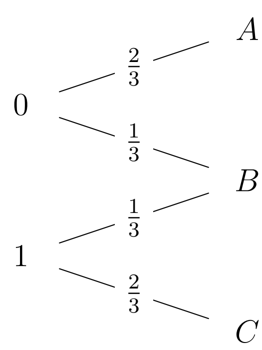
$$P(|X_n| > \epsilon) = P(X_n \neq 0) = \frac{1}{n^2} \to 0$$

However, the variance does not go to zero. Instead, it converges to 1.

$$\begin{aligned} \text{Var}(X_n) &= \mathbb{E}[X_n^2] - \mathbb{E}[X_n]^2 \\ &= 1 - \frac{1}{n^2} \to 1 \end{aligned}$$

(b) **Channel Capacity**

Use the formula $C = \max_{p_X} H(Y) - H(Y \mid X)$ to find the capacity of the following channel. You may write your answer in terms of $H_b(p) = -p \log_2 p - (1-p)\log_2(1-p)$.



Let $P(X=1) = p$. $H(Y \mid X=0) = H(Y \mid X=1) = H_b(\frac{2}{3})$, so $H(Y \mid X) = H_b(\frac{2}{3})$ no matter what p is. Then, to find the capacity, it just remains to maximize $H(Y) - H_b(\frac{2}{3})$. We know that the uniform distribution maximizes entropy, and letting $p = \frac{1}{2}$, $P(Y=A) = P(Y=B) = P(Y=C) = \frac{1}{3}$. So the answer is $\log_2(3) - H_b(\frac{1}{3})$, which simplifies to $\frac{2}{3}$.

Alternate Solution

Another way to solve this problem is to recognize that it is essentially a BEC with $p = \frac{1}{3}$.

(c) **Symbols**

We would like to encode the symbols A, B, C, D with some bit sequences. We know that their probabilities are 0.1, 0.2, 0.3, and 0.4, respectively.

(i) What is the entropy of the distribution of symbols? You may leave your answer as an expression.

Based on the formula for entropy, it is,

$$-0.1\log(0.1) - 0.2\log(0.2) - 0.3\log(0.3) - 0.4\log(0.4) \approx 1.85$$

(ii) Under the optimal encoding procedure, what is the average length of an encoded symbol?

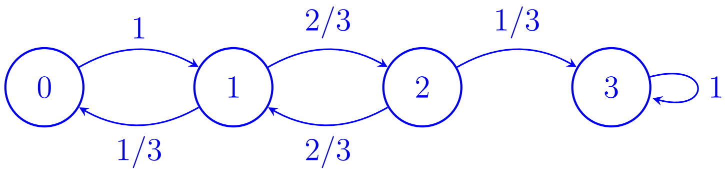
The optimal encoding procedure is Huffman encoding. First, A and B will be combined. Then (A, B) and C will be combined. Finally, (A, B, C) and D will be combined. A's bit sequence will be 000, B's will be 001, C's will be 01, and D's will be 1. This leads to an average length of

$$0.1 \cdot 3 + 0.2 \cdot 3 + 0.3 \cdot 2 + 0.4 \cdot 1 = 0.3 + 0.6 + 0.6 + 0.4 = 1.9$$

(d) **Chair Game**

Suppose there are three students arranged in a circle. Initially, at time $t = 0$, all three students are sitting down. Every second, a student is chosen uniformly at random. If this student was originally sitting down, then they would stand up, and if they were standing up, they now sit down. What is the expected amount of time that passes until the first time in which all three students are standing up? Leave your answer as a simplified fraction.

Let us create a Markov chain based on the number of students currently standing up.



When x people are standing up, the probability of transitioning to state $x - 1$ is $\frac{x}{3}$, because out of the 3 people in total, selecting x of them will result in one of them sitting down, thus transitioning to $x - 1$. Similarly, the probability of transitioning to state $x + 1$ is $\frac{3-x}{3}$. We can now set up the first step equations to get the expected hitting time from state 0 to state 3.

$$\begin{aligned}
E_0 &= 1 + E_1 \\
E_1 &= 1 + \frac{1}{3}E_0 + \frac{2}{3}E_2 \\
E_2 &= 1 + \frac{2}{3}E_1 + \frac{1}{3}E_3 \\
E_3 &= 0
\end{aligned}$$

To solve this system, first get that $E_2 = 1 + \frac{2}{3}E_1$, then take the following steps.

$$\begin{aligned}
E_1 &= 1 + \frac{1}{3}E_0 + \frac{2}{3}(1 + \frac{2}{3}E_1) \\
E_1 &= \frac{5}{3} + \frac{1}{3}E_0 + \frac{4}{9}E_1 \\
E_1 &= \frac{9}{5}(\frac{5}{3} + \frac{1}{3}E_0) \\
E_1 &= 3 + \frac{3}{5}E_0 \\
E_0 &= 1 + 3 + \frac{3}{5}E_0 \\
E_0 &= \frac{5}{2} \cdot 4
\end{aligned}$$

The answer is 10.

(e) **Running Track**

Avishek is pacing around his school running track, which can broken up into 400 1-meter segments. Every time step, he decides with equal probability either to go 1-meter forward, or to turn around and go 1-meter backwards. Given he is currently somewhere along the track, how long will it take for him to next return to his initial position?

This problem can be modeled with a Markov chain with states $\{1, 2, \ldots, 400\}$ written around a ring. At every time step, you jump with equal probability to one of the adjacent numbers. Since this Markov chain is irreducible, the expected return time from state x to itself is $1/\pi(x)$, where π is the stationary distribution. By symmetry, the stationary distribution is $1/400$ at every state, so the answer is 400.

Alternate Problem Solution

Some people interpreted the track to be a straight line. In this case, the formula $1/\pi(x)$ still applies. $\pi(x)$ can be calculated by realizing that the Markov chain represents an undirected graph and using the formula $\pi(x) = \frac{deg(x)}{2E}$. There are 399 edges, so if we assume Avishek starts at an endpoint, $\pi(x) = \frac{1}{2 \cdot 399}$, otherwise $\pi(x) = \frac{2}{2 \cdot 399}$. So the answer would be 798 or 399, respectively.

(f) **Tipsy Bartenders**

Two bartenders continuously pour drinks a rate of 1L/min into their own respective glasses and serve them according to Poisson Processes with rates 2 drinks/min and 3 drinks/min. Say this process started infinitely long in the past. Given you now choose a bartender **randomly with equal probability**, and you take the next drink the chosen bartender serves, what is the expected volume of your drink?

Since we are observing the Poisson Processes infinitely past its initialization, we employ the result of the Random Incidence Paradox, which states that the length of an interval we choose randomly will be distributed as Erlang$(2, \lambda)$. Thus, the average length of an interval we choose will be twice the average length of an interarrival time. Since we choose the bartender we observe uniformly at random, we take a weighted combination of their expectations.

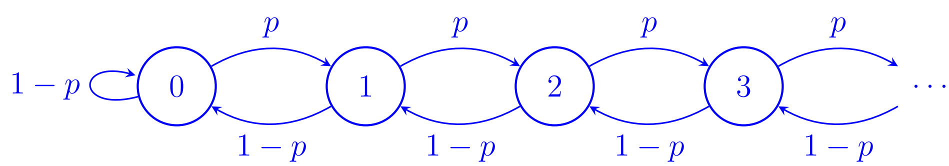
Let the volume of your drink be represented by D, and the choice of bartender is represented by $B \in 1, 2$.

$$\begin{aligned}\mathbb{E}[D] &= \mathbb{E}[D|B=1] * P(B=1) + \mathbb{E}[D|B=2] * P(B=2) \\ &= 1 * \frac{1}{2} + \frac{2}{3} * \frac{1}{2} = \frac{5}{6}\text{L}\end{aligned}$$

2 Tacos

A popular restaurant is serving tacos. Each time step, **exactly one** of the following two things occur. With probability p, a new customer comes into the restaurant. With probability $1-p$, a customer waiting in line is served (if there are no customers waiting in line, nothing happens).

The following Markov chain represents the number of people in the restaurant.



(a) Suppose customers are currently in the restaurant. The restaurant cooks like making tacos, but they also want to take a break eventually (when there are no customers in the store). In order for this happen with probability 1, what must be true of p? Briefly justify.

For us to be able to return to state 0 eventually, the chain needs to be recurrent, and the requirement on that is that $p \le \frac{1}{2}$.

(b) However, the cooks also don't want to wait forever for their break. In order for the expected amount of time they have to wait to be finite, what must be true of p? Briefly justify.

For us to be able to return to state 0 in finite time, the chain needs to be positive recurrent, and the requirement on that is that $p < \frac{1}{2}$.

(c) Suppose $p = 1/3$. Sometime in the distant future, Will arrives at the restaurant. What is the probability the restaurant is empty?

In the distant future, the number of people in the restaurant will following the stationary distribution of the Markov chain. The stationary distribution exists since the chain is positive recurrent ($\frac{1}{3} < \frac{1}{2}$) and can be calculated through the DBE $\pi(i)p = \pi(i+1)(1-p)$. This implies that $\pi(i+1) = \frac{p}{1-p}\pi(i) = \frac{1}{2}\pi(i)$.

$$1 = \sum_{i=0}^{\infty} \pi(i) = \sum_{i=0}^{\infty} (\frac{1}{2})^i \pi(0) = \pi(0) \sum_{i=0}^{\infty} (\frac{1}{2})^i = \pi(0) \frac{1}{1-\frac{1}{2}} = 2\pi(0)$$

This implies $\pi(0) = \frac{1}{2}$.

3 Basketball

Kevin and Nikita are playing basketball. Baskets are made according to a Poisson process with parameter $\lambda = 2$. Each basket independently has probability $\frac{1}{3}$ of being from Kevin and probability $\frac{2}{3}$ of being from Nikita.

(a) At time $t = 3$, what is the expected lead (in baskets made) that Kevin has over Nikita? This can be negative.

> Each basket made leads to an expected lead of $\frac{1}{3} - \frac{2}{3} = -\frac{1}{3}$ for Kevin. The number of baskets in $[0, t]$ is distributed as Poisson(λt), so the expectation is $-\frac{1}{3}\lambda t = -\frac{1}{3} \cdot 2 \cdot 3 = -2$.

(b) Suppose at time $t = 30$, Kevin had made seven baskets. What is the expected number of baskets Nikita made when Kevin just makes his fifth basket?

> Say Kevin makes his fifth basket at time T_5. Since Kevin and Nikita are split Poisson processes, Nikita's baskets in $[0, T_5]$ is distributed as Poisson($\frac{2}{3}\lambda T_5$), leading to an expectation of $\frac{2}{3}\lambda T_5$. However, conditioning on the fact that Kevin scored seven baskets from $[0, t]$, T_5 is the 5th order statistic of 7 i.i.d. Uniform$[0, t]$ random variables. Thus the answer is
>
> $$\mathrm{E}[\frac{2}{3}\lambda T_5] = \frac{2}{3}\lambda \cdot \frac{5}{7+1} t = \frac{4}{3} \cdot \frac{5}{8} \cdot 30 = 25$$

(c) Suppose at $t = 4$, three baskets have been made. What is the probability that exactly two baskets were made from $t = 0$ to $t = 2$?

> The times of the 3 baskets T_1, T_2, T_3 are the order statistics of 3 i.i.d. Uniform$[0, 4]$ random variables. The problem is asking for $P(T_1, T_2 \in [0, 2], T_3 \in [2, 4])$ given our situation. If we let U_1, U_2, U_3 be the underlying RVs, this probability is
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> $$P(U_1, U_2 \in [0, 2], U_3 \in [2, 4]) + P(U_1, U_3 \in [0, 2], U_2 \in [2, 4]) + P(U_2, U_3 \in [0, 2], U_1 \in [2, 4])$$
>
> which is $3 \cdot (\frac{1}{2})^3 = \frac{3}{8}$.

(d) Suppose Kevin only goes for 3-pointers and Nikita only goes for 2-point layups. At time $t = 3$, the **total points scored** is currently 8. Given this, what is the probability that Kevin scored 6 points?

If the current score is 8, then the only possible outcomes are that Kevin made 2 baskets and Nikita made 1 or Nikita made all 4. Let $N_K(t)$ be the number of baskets Kevin makes in $[0, t]$, and similarly let $N_N(t)$ be the number of baskets Nikita makes in $[0, t]$. Note that they correspond to a split Poisson process, so they are independent. We want to find

$$\frac{\mathrm{P}(N_K(t) = 2)\,\mathrm{P}(N_N(t) = 1)}{\mathrm{P}(N_K(t) = 2)\,\mathrm{P}(N_N(t) = 1) + \mathrm{P}(N_K(t) = 0)\,\mathrm{P}(N_N(t) = 4)}$$

We can write out the individual products

$$\mathrm{P}(N_K(t) = 2)\,\mathrm{P}(N_N(t) = 1) = \frac{e^{-\frac{1}{3}\lambda t}(\frac{1}{3}\lambda t)^2}{2!} \cdot \frac{e^{-\frac{2}{3}\lambda t}(\frac{2}{3}\lambda t)^1}{1!}$$
$$\mathrm{P}(N_K(t) = 0)\,\mathrm{P}(N_N(t) = 4) = \frac{e^{-\frac{1}{3}\lambda t}(\frac{1}{3}\lambda t)^0}{0!} \cdot \frac{e^{-\frac{2}{3}\lambda t}(\frac{2}{3}\lambda t)^4}{4!}$$

Thankfully the e terms cancel out. Since $\lambda t = 2 \cdot 3 = 6$, the answer is

$$\frac{\frac{2^2}{2} \cdot 4}{\frac{2^2}{2} \cdot 4 + \frac{4^4}{24}} = \frac{8}{8 + \frac{64}{6}} = \frac{48}{48 + 64} = \frac{3}{7}$$