problem # 1.

(a) Apply energy conservation.

$$\int_0^L \dot{q}(x)\,dx = h\,(T(L) - T_\infty)$$

$$\Rightarrow \int_0^L ax\,dx = \left[\frac{1}{2}ax^2\right]_0^L = \frac{1}{2}aL^2.$$

$$\therefore \frac{1}{2}aL^2 = h\,(T(L) - T_\infty)$$

$$T(L) = T_\infty + \frac{aL^2}{2h}.$$

Different approach;

Heat equation.

$$\frac{\partial^2 T}{\partial x^2} + \frac{a}{k}x = 0.$$

Boundary condition.

$$\begin{cases} \left.\dfrac{\partial T}{\partial x}\right|_{x=0} = 0. & \cdots ① \\ -k\left.\dfrac{\partial T}{\partial x}\right|_{x=L} = h\,(T(L) - T_\infty) & ② \end{cases}$$

$$T(x) = -\frac{a}{6k}x^3 + C_1 x + C_2.$$

Apply B.C ①. $\Rightarrow C_1 = 0.$

$$\therefore T(x) = -\frac{a}{6k}x^3 + C_2 \quad \cdots ④$$

Apply B.C ③.

$$\frac{a}{2}L^2 = h\,(T_{(L)} - T_\infty)$$

$$T_{(L)} = T_\infty + \frac{aL^2}{2h}.$$

(b) From ④

$$T_{(L)} = -\frac{a}{6k}L^3 + C_2 = T_\infty + \frac{aL^2}{2h}.$$

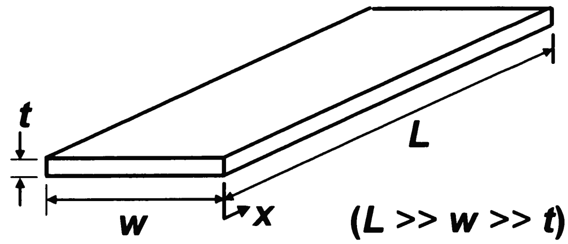
$$C_2 = T_\infty + \frac{a}{6k}L^3 + \frac{aL^2}{2h}.$$

$$\therefore\ T_{(x)} = -\frac{a}{6k}x^3 + T_\infty + \frac{a}{6k}L^3 + \frac{aL^2}{2h}.$$

$$T_{(0)} = T_\infty + \frac{a}{6k}L^3 + \frac{aL^2}{2h}.$$

Problem 2. (20 pts)

Consider the fin depicted in the figure, with a standard rectangular ($w \times t$) cross section. The base at x=0 is held at T_{base}. The fin length L is long enough to be approximated as infinitely long for heat transfer purposes.



You should assume $L >> w >> t$ throughout this problem.

This fin is being revised, from the initial (1) to new (2) design. Your task is to specify the new t_2 and w_2 which satisfy the following (also summarized in the table below):

- h and L remain fixed for both designs,
- The new design uses a 3-fold higher thermal conductivity.
- You must ensure that both designs exhibit the same temperature profile $T(x)$.
- You must double the overall fin heat rate, q [in Watts].

	t	w	k	$T(x)$	Fin q	L, h, T_{base}, T_∞
Initial Design	t_1	w_1	k_1	$T_1(x)$	q_1	(no change; same for both cases)
New Design	***find*** t_2	***find*** w_2	$k_2 = 3*k_1$	Design for no change, $T_2(x) = T_1(x)$	Design for $q_2 = 2*q_1$	

(a:10 pts) Find an expression for the new t_2 which satisfies the above requirements.

(b:10 pts) Find an expression for the new w_2 which satisfies the above requirements.

FOR INFINITE FIN : $\frac{\theta}{\theta_b} = e^{-mx}$; $q_F = M$

a) SINCE $T_2(x) = T_1(x)$, $\theta_1(x) = \theta_2(x) \Rightarrow m_1 = m_2$ $\qquad m \equiv \sqrt{\frac{hP}{kA_c}}$

SO $\left(\frac{hP}{kA_c}\right)_1 = \left(\frac{hP}{kA_c}\right)_2$ $\cdots [k_2 = 3k_1] \cdots\rightarrow$ $\frac{P_1}{A_{c1}} = \frac{P_2}{3A_{c2}}$

KNOW $P = 2w + \underbrace{\cancel{2t}}_{t \ll w} \approx 2w$, $A_c = wt$, SO $\frac{2w_1}{w_1 t_1} = \frac{2w_2}{3w_2 t_2}$ $\Rightarrow \boxed{t_2 = \frac{t_1}{3}}$

b) SINCE $q_2 = 2q_1$ $\quad M_2 = 2M_1$ OR $\left(\sqrt{hPkA_c}\,\theta_b\right)_2 = 2\left(\sqrt{hPkA_c}\,\theta_b\right)_1$

$\Rightarrow P_1 A_{c1} = 2^2 \cdot 3 P_2 A_{c2}$ $\Rightarrow 3(2w_2)(w_2 t_2) = 2^2(2w_1)(w_1 t_1)$

OR $3w_2^2 \frac{t_1}{3} = 4w_1^2 t_1$ $\Rightarrow w_2^2 = 4w_1^2$

$\Rightarrow \boxed{w_2 = 2w_1}$

problem 3.

(a).

$$B_i = \frac{h L_c}{k}$$

$$L_c = \frac{V}{A_s} = \frac{L}{6} \quad \left(L_c = \frac{L}{2}, \frac{\sqrt{3}}{2}L \text{ also o.k} \right).$$

$$\therefore B_i = \frac{hL}{6k}$$

$$B_{i\,Al} = \frac{h \times 0.01}{6 k_{Al}} = \frac{h \times 0.01}{1200} < 0.1$$

$$h < 12000.$$

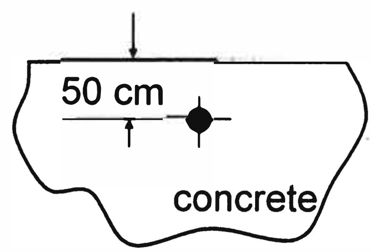
Problem 3: Short Answers (10 pts)

(a: 5 pts). Recall the demo from Lecture 1 when we immersed a small cube of aluminum (T_i=300 K) into liquid nitrogen (T_∞ = 77 K). Take the cube edge length to be L = 1 cm. We would like to model this heat transfer using a lumped capacitance treatment.

Properties for Problem 3	*k* [W/m-K]	ρ [kg/m³]	*c* [J/kg-K]
Aluminium	200	3,000	1,000
Concrete	1.0	2,000	1,000

Calculate the range of convection coefficient *h*, for liquid nitrogen boiling, for which a lumped treatment would be reasonable.

(b: 5 pts). An aluminum cylinder of diameter 10 cm is very long in the direction out of the page. The cylinder is embedded at a depth of 50 cm in a very large and thick slab of concrete. The top of the slab is at T_1=50 °C, and the cylinder is at T_2=10 °C.



Calculate the heat transfer into the cylinder, per unit length.

USE 1^{ST} LISTED SHAPE FACTOR

SINCE $L = \infty \Rightarrow L >> D$

AND $z = 50\,cm \quad D = 10\,cm \Rightarrow z > \frac{3D}{2}$ ✓

$\Rightarrow S = \frac{2\pi L}{\ln\left(\frac{4z}{D}\right)}$

$$q = Sk(T_1 - T_2) = \frac{2\pi L k}{\ln\left(\frac{4z}{D}\right)}(T_1 - T_2)$$

$$\frac{q}{L} = \frac{2\pi k}{\ln\left(\frac{4z}{D}\right)}(T_1 - T_2) = \frac{2\pi\,(1\,W/m\cdot K)}{\ln\left(\frac{4\cdot 0.5m}{0.1m}\right)}(50-10)°C = \boxed{83.9\ \text{W/m} = \frac{q}{L}}$$