EECS 126 Probability and Random Processes — UC Berkeley, Spring 2020
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Midterm 1

Last Name	First Name	SID

Left Neighbor First and Last Name	**Right Neighbor** First and Last Name

***Rules*.**

- **Unless otherwise stated, all your answers need to be justified.**
- You have 10 minutes to read the exam and 90 minutes to complete it.
- The exam is not open book. No calculators or phones allowed.
- Write in your SID on every page to receive 1 point.

Problem	points earned	out of
Problem 1		62
Problem 2		12
Problem 3		12
Problem 4		18
SID		1
Total		105

1 Assorted Problems (62 points)

(a) **True / False (6 points)**

Suppose $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$ are exponentially distributed random variables and are independent. Then they are uncorrelated. Justify your answer.

◯ True ◯ False

True. Independence implies that $\mathrm{E}[XY] = \mathrm{E}[X]\,\mathrm{E}[Y]$, which implies $\mathrm{cov}(X, Y) = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y] = 0$.

Suppose again $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$ are exponentially distributed random variables and are independent. Then their sum is exponentially distributed. Assume, $\lambda > \mu$, and justify your answer.

◯ True ◯ False

False. Convolving their densities,

$$
\begin{aligned}
f_{X+Y}(z) &= \int_0^z \lambda e^{-\lambda x} \mu e^{-\mu(z-x)}\,dx \\
&= \frac{\lambda\mu}{\lambda-\mu} e^{-\mu z} \int_0^z (\lambda - w\mu) e^{-(\lambda-\mu)x}\,dx \\
&= \frac{\lambda\mu}{\lambda-\mu} e^{-\mu z}
\end{aligned}
$$

This doesn't have the form of an exponential PDF. Note: It was also possible to show this by multiplying their MGFs.

(b) **Drawing Balls (6 points)**

Alan and 3 of his friends are drawing balls out of a bag. In the bag, there are 3 red balls, 4 white balls, and 5 blue balls. After his 3 friends draw without replacement, Alan takes his turn. What's the probability that he draws a blue ball?

In this experiment, we can consider the outcomes to be all 12! permutations of the balls, and the balls that Alan and his 3 friends draw to be the 4th, 1st, 2nd, and 3rd ball in this permutation, respectively. The probability the 4th ball is blue is no different from the probability the 1st ball is blue, so $\frac{5}{12}$.

(c) **Covariance (6 points)**

Suppose $X, Y \sim \text{Uniform}[-\frac{1}{2}, \frac{1}{2}]$, and $\text{var}(X - Y) = 1$. What is $\text{cov}(X, Y)$?

$$\begin{aligned} \text{var}(X - Y) &= \text{cov}(X - Y, X - Y) \\ &= \text{cov}(X, X) - 2\,\text{cov}(X, Y) + \text{cov}(Y, Y) \\ &= 2\,\text{var}(X) - 2\,\text{cov}(X, Y) \end{aligned}$$

Since $X \sim U[-\frac{1}{2}, \frac{1}{2}]$, $\text{var}(X) = \text{var}(Y) = \frac{(\frac{1}{2} - (-\frac{1}{2}))^2}{12} = \frac{1}{12}$, and therefore,

$$\begin{aligned} \text{cov}(X, Y) &= \frac{2\,\text{var}(X) - \text{var}(X - Y)}{2} \\ &= \frac{\frac{2}{12} - 1}{2} = -\frac{5}{12} \end{aligned}$$

(d) **Exponential Race (6 points)**

Alice and Bob are serving customers at a store counter. Their service times are modeled as independent exponential random variables, with Alice's rate being twice that of Bob's. Suppose you arrive at the counter to find that Alice and Bob are both busy with customers, and there are no other customers in the store. You will be served as soon as either Alice or Bob is free. What is the probability that you will be the last customer to leave the store?

The probability that Alice services her customer first is $\frac{2\lambda}{2\lambda + \lambda} = \frac{2}{3}$. Now being served by Alice, the probability that you finish after Bob's customer is $\frac{1}{3}$ by the memoryless probability. Since Bob could have serviced his customer first, the total probability is $\frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9}$.

(e) **Fountain Codes (6 points)**

You have 10 chunks to decode using fountain codes. You have three 2-degree packets, five 3-degree packets, one 4-degree packet and one 1-degree packet. All the chunks in the packets are drawn uniformly at random. What is the expected number of 3-degree packets after peeling the 1-degree packet off?

As an example, if a 3-degree packet contains the chunk in the 1-degree packet, after peeling the 1-degree packet off, it will become a 2-degree packet.

$$\begin{aligned}\text{E[\# 3-degree packets]} &= 5 * \text{E[ith 3-degree packet not peeled]} + \text{E[ith 4-degree packet peeled]} \\ &= 5 * \frac{7}{10} + \frac{4}{10} = \frac{39}{10}\end{aligned}$$

(f) **Throwing Axe (6 points)**

You go axe-throwing with your friends and you noticed an axe sticking out of the board from the previous session. $\frac{1}{4}$ of axe throwers are left-handed. and $\frac{3}{4}$ are right-handed. Let X be a continuous random variable representing where an axe lands on the horizontal axis relative to the middle. You know that left-handed throwers land their axe according to a $X \sim N(-1, 1)$ distribution. Right-handed throwers land their axe according to a $X \sim N(1, 1)$ distribution. Given that you see an axe sticking out of the board at $X = \frac{-1}{2}$, what is the probability that the person who threw the axe was a right-handed thrower?

Let R be an indicator RV which takes the value 1 when the person is right handed and 0 otherwise.

$$\begin{aligned}P(R = 1|X = \frac{-1}{2}) &= \frac{f(X = \frac{-1}{2}|R = 1)P(R = 1)}{P(X = \frac{-1}{2}|R = 1)P(R = 1) + P(X = \frac{-1}{2}|R = 0)P(R = 0)} \\ &= \frac{\frac{1}{\sqrt{2\pi}}e^{-(-\frac{1}{2}-1)^2/2} \cdot \frac{3}{4}}{\frac{1}{\sqrt{2\pi}}e^{-(-\frac{1}{2}-1)^2/2} \cdot \frac{3}{4} + \frac{1}{\sqrt{2\pi}}e^{-(-\frac{1}{2}-(-1))^2/2} \cdot \frac{1}{4}} \\ &= \frac{\frac{3}{4}e^{-9/8}}{\frac{3}{4}e^{-9/8} + \frac{1}{4}e^{-1/8}} \\ &= \frac{3}{3+e} \approx 0.52\end{aligned}$$

(g) **Telephone (6 points)**

Let $X_1 \sim \mathcal{N}(0,1)$ and $X_i \sim \mathcal{N}(X_{i-1},1)$. What is the distribution of X_n?

$$\begin{aligned} X_n \sim \mathcal{N}(X_{n-1},1) &= X_{n-1} + \mathcal{N}(0,1) \\ &= \sum_{i=1}^{n} \mathcal{N}(0,1) \\ &= \mathcal{N}(\sum_{i=1}^{n} 0, \sum_{i=1}^{n} 1) \\ &= \mathcal{N}(0,n) \end{aligned}$$

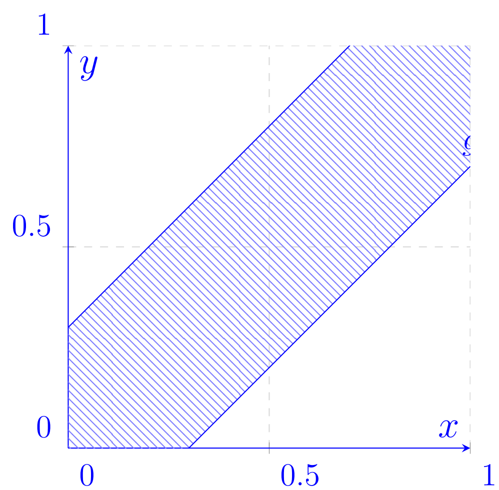
(h) **Cutting the Rope (6 points)**

Consider a rope with length 1. One uniformly and independently picks two places on the rope and cuts the rope to three pieces. Denote Z as the length of the middle piece. Find the CDF of Z and $\mathrm{E}(Z)$.

Consider X, Y are independent and uniform on $[0,1]$, then we have $Z = \max(X,Y) - \min(X,Y) = |X-Y|$.

$$P(Z \le t) = P(|X-Y| \le t) = \iint_{|r-s|\le t, 0\le r,s\le 1} 1 drds = 1-(1-t)^2.$$

The integral is equivalent to the shaded area in the plot below. To see this, note that for given $X = x$, we have that $|X-Y| \le t$ means that $x-t \le Y \le x+t$. The shaded area is equal to the area of the unit square minus the sum of the areas of the triangles. The triangles have side lengths $1-t$, so the sum of their areas is $(1-t)^2$.



(i) **MGF for Binomial (6 points)**

For $Y \sim \text{Binom}(n, 1/2)$, show that the MGF of $Y - \mathrm{E}[Y]$ is at most $e^{ns^2/8}$.
Hint: You can use the fact that $\frac{1}{2}(e^{s/2} + e^{-s/2}) \leq e^{s^2/8}$.

We can write the binomial as the sum of independent centered Bernoullis. For $X \sim \text{Bernoulli}(1/2)$, the MGF of $X - \mathrm{E}[X]$ can be found either directly (it takes on values $1/2$ and $-1/2$ with equal probability) or through the MGF of X.

$$\mathrm{E}[e^{s(X-\mathrm{E}[X])}] = \frac{1}{2}(e^{s/2} + e^{-s/2}) \leq e^{s^2/8},$$

where the bound follows from part (a). Then, the MGFs multiply by independence:

$$\begin{aligned}
\mathrm{E}[e^{s(Y-\mathrm{E}[Y])}] &= \mathrm{E}[e^{s\sum_{i=1}^{n}(X_i-\mathrm{E}[X_i])}] \\
&= \prod_{i=1}^{n} e^{s(X_i-\mathrm{E}[X_i])} \\
&\leq \prod_{i=1}^{n} e^{s^2/8} \\
&= e^{ns^2/8}.
\end{aligned}$$

(j) **Tight Inequalities (8 points)**

(i) (4 points) The Markov bound is generally quite loose, but for certain random variables and certain values it can be tight. For a fixed k, construct a random variable X that assumes only non-negative integer values such that $P(X \geq k) = \frac{\mathrm{E}[X]}{k}$.

Consider $Y = \text{Bernoulli}(\frac{1}{k})$ and $X = kY$. Then $P(X \geq k) = P(X = k) = \frac{1}{k} = \frac{k \cdot \frac{1}{k}}{k} = \frac{\mathrm{E}[X]}{k}$.

(ii) (4 points) For a fixed k, construct a random variable X that is tight for the Chebyshev inequality, or show that this is impossible. Justify your answer.

Let

$$X = \begin{cases} k & \text{w.p. } 1/2 \\ -k & \text{w.p. } 1/2 \end{cases}$$

Then $P(|X| \geq k) = 1 = \frac{k^2 - 0^2}{k^2} = \frac{\mathrm{E}[X^2] - \mathrm{E}[X]^2}{k^2} = \frac{\mathrm{var}(X)}{k^2}$.

2 Good and Bad Items (12 points)

Suppose you have N items, G of which are good and B of which are bad. You start to draw items without replacement, and suppose that the first good item appears on draw X.

This question appeared on Discussion 2.

(a) (6 points) What is $\mathrm{E}[X]$?

The expectation is computed with a clever trick: let X_i be the indicator if bad item i appears before the first good item. Here, we do not refer to the i-th bad item drawn but simply use i to index each bad item. Then,

$$X = 1 + \sum_{i=1}^{B} X_i$$

and by linearity of expectation,

$$\mathrm{E}[X] = 1 + B\,\mathrm{E}[X_1] = 1 + \frac{B}{G+1} = \frac{N+1}{G+1}.$$

(b) (6 points) What is $\operatorname{var}(X)$? No need to simplify your answer.

Observe that $\operatorname{var} X = \operatorname{var}(X-1)$. Using the same indicators, we compute $\mathrm{E}[(X-1)^2]$.

$$\begin{aligned}\mathrm{E}[(X-1)^2] &= B\,\mathrm{E}[X_1^2] + B(B-1)\,\mathrm{E}[X_1X_2] \\ &= \frac{B}{G+1} + \frac{2B(B-1)}{(G+1)(G+2)}\end{aligned}$$

Therefore, our answer is

$$\operatorname{var} X = \frac{B}{G+1} + \frac{2B(B-1)}{(G+1)(G+2)} - \left(\frac{B}{G+1}\right)^2.$$

With a little algebra, we can actually simplify the result.

$$\begin{aligned}\operatorname{var} X &= \frac{B(G+1)(G+2) + 2B(B-1)(G+1) - B^2(G+2)}{(G+1)^2(G+2)} \\ &= \frac{BG(N+1)}{(G+1)^2(G+2)}\end{aligned}$$

3 Urns and Balls (12 points)

Two urns contain a large number of balls with each ball marked with one number from the set $\{0, 2, 4\}$. The proportion of each type of ball in each urn is displayed in the table below:

		Number on Ball X		
		0	2	4
Urn Label θ	A	0.6	0.3	0.1
	B	0.1	0.3	0.6

An urn is randomly selected with equal probability and then a ball is drawn at random from the urn. The urn from which the ball is selected (either A or B) is represented by θ and the number on the ball is represented by X.

(a) (5 points) Find $\text{var}(\text{E}[X|\theta])$ (place this value in box).

$$\text{E}[X \mid \theta = A] = 0 \cdot 0.6 + 2 \cdot 0.3 + 4 \cdot 0.1 = 1$$
$$\text{E}[X \mid \theta = B] = 0 \cdot 0.1 + 2 \cdot 0.3 + 4 \cdot 0.6 = 3$$

So

$$\text{E}[X \mid \theta] = \begin{cases} 1 & \text{w.p. } 1/2 \\ 3 & \text{w.p. } 1/2 \end{cases}$$

and

$$\begin{aligned} \text{var}(\text{E}[X \mid \theta]) &= \text{E}[\text{E}[X \mid \theta]^2] - \text{E}[\text{E}[X \mid \theta]]^2 \\ &= \Big(1 \cdot 0.5 + 9 \cdot 0.5\Big) - \Big(1 \cdot 0.5 + 3 \cdot 0.5\Big)^2 \\ &= 5 - 4 = 1 \end{aligned}$$

(b) (5 points) Find $\text{E}[\text{var}(X|\theta)]$ (place this value in box).

We know $\text{var}(X \mid \theta) = \text{E}[X^2 \mid \theta) - \text{E}[X \mid \theta]^2$

$$\begin{aligned} \text{var}(X \mid \theta = A) &= \text{E}[X^2 \mid \theta = A] - 1^2 \\ &= \Big(0 \cdot 0.6 + 4 \cdot 0.3 + 16 \cdot 0.1\Big) - 1 \\ &= 1.8 \\ \text{var}(X \mid \theta = B) &= \text{E}[X^2 \mid \theta = B] - 3^2 \\ &= \Big(0 \cdot 0.1 + 4 \cdot 0.3 + 16 \cdot 0.6\Big) - 9 \\ &= 1.8 \end{aligned}$$

So

$$\operatorname{var}(X \mid \theta) = \begin{cases} 1.8 & \text{w.p. } 1/2 \\ 1.8 & \text{w.p. } 1/2 \end{cases}$$

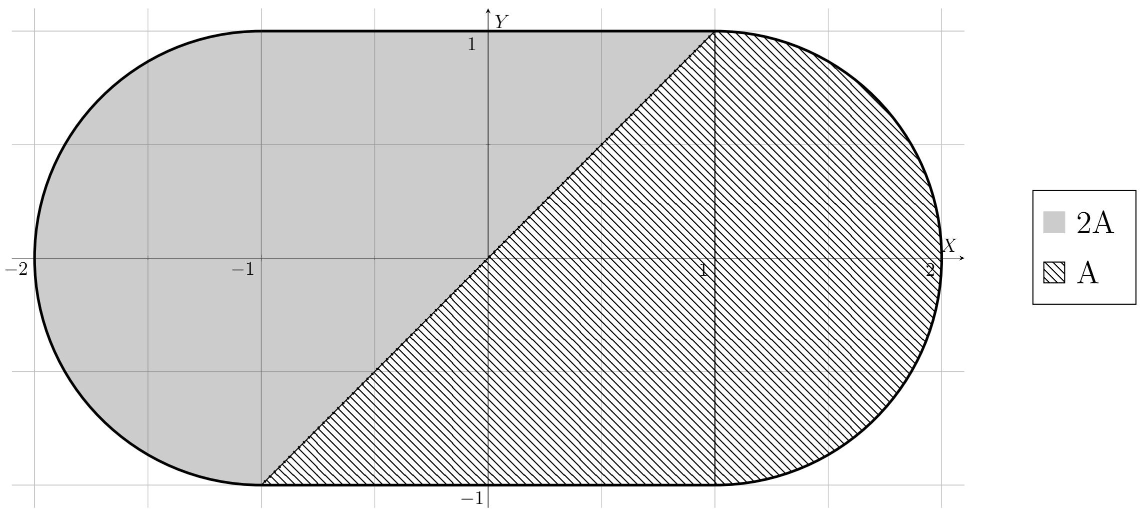
So $\mathrm{E}[\operatorname{var}(X \mid \theta)] = 1.8$

(c) (2 points) Calculate $\operatorname{var}(X)$.

By the law of total variance, $\operatorname{var}(X) = \operatorname{var}(\mathrm{E}[X \mid \theta]) + \mathrm{E}[\operatorname{var}(X \mid \theta)] = 1 + 1.8 = 2.8$.

4 Graphical Density (18 points)

Random variables X and Y have a joint density as pictured below:



(a) (5 points) What is A?

$$\frac{\pi}{2} \cdot 2A + 2 \cdot 2A + 2 \cdot A + \frac{\pi}{2} \cdot A = 1$$

This implies $(\frac{3\pi}{2} + 6)A = 1$, so $A = \frac{1}{\frac{3\pi}{2}+6}$.

(b) (5 points) In terms of A, what is the marginal density of X?
Hint: The equation of a circle of radius 1 centered at (a, b) is $(x - a)^2 + (y - b)^2 = 1$.

For a given x, $f_X(x) = \int f_{X,Y}(x, y)\, dy$. We can break this down into the following cases.

- When $x \in [-2, -1]$, the integral becomes

$$\int_{-\sqrt{1-(x-(-1))^2}}^{\sqrt{1-(x-(-1))^2}} 2A\, dy = 4A\sqrt{1-(x+1)^2}$$

- When $x \in [-1, 1]$, the integral becomes

$$\int_{-1}^{x} A\, dy + \int_{x}^{1} 2A\, dy = A(x+1) + 2A(1-x) = (3-x)A$$

- Similar to $x \in [-2, -1]$, when $x \in [1, 2]$ the integral becomes $2A\sqrt{1-(x-1)^2}$

(c) (5 points) Suppose $\mathrm{E}[X \mid X \in [1,2]] = 1 + \frac{4}{3\pi}$. Let Z be an indicator if (X, Y) is in the region with density A. What is $\mathrm{E}[X \mid Z = 1]$? You don't need to simplify your answer.

- Conditioned on being in the triangle portion, the expectation of X is $\frac{2}{3}$ "of the way", meaning it is $-1 + (1 - (-1))\frac{2}{3} = \frac{1}{3}$.
- As mentioned in the hint, the expectation in the semicircle portion is $1 + \frac{4}{3\pi}$.
- The area of the triangle is 2, and the area of the semicircle is $\frac{\pi}{2}$.

So

$$\begin{aligned}
\mathrm{E}[X \mid Z = 1] &= (1 + \frac{4}{3\pi}) \cdot \frac{\frac{\pi}{2}}{2 + \frac{\pi}{2}} + \frac{1}{3} \cdot \frac{2}{2 + \frac{\pi}{2}} \\
&= \Big(\frac{3\pi + 4}{3\pi}\Big)\Big(\frac{\pi}{4 + \pi}\Big) + \frac{1}{3}\Big(\frac{4}{4 + \pi}\Big) \\
&= \frac{3\pi + 4}{3\pi + 12} + \frac{4}{3\pi + 12} \\
&= \frac{3\pi + 8}{3\pi + 12}
\end{aligned}$$

(d) (3 points) Suppose the answer to the previous question is a. In terms of that, what is $\mathrm{E}[X]$?

By symmetry, $\mathrm{E}[X \mid Z = 0] = -a$. Since $\mathrm{E}[X] = \mathrm{E}[\mathrm{E}[X \mid Z]]$ and the $2A$ region has twice the mass as the A region,

$$\mathrm{E}[X] = \frac{2}{3}(-a) + \frac{1}{3}a = -\frac{a}{3}$$