EE 120: Signals and Systems
Department of Electrical Engineering and Computer Sciences — MIDTERM 2
UC BERKELEY — 21 March 2019

FIRST Name ______________ LAST Name ______________

Discussion Section Time: __________ SID (All Digits): __________

- **(10 Points)** Print your *official* name (not your e-mail address) and *all* digits of your student ID number legibly, and indicate your lab time, on *every* page.
- This exam should take up to 100 minutes to complete. However, you may use up to a maximum of 110 minutes *in one sitting*, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except two double-sided $8.5" \times 11"$ sheets of handwritten, original notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- We will provide you with scratch paper. Do not use your own.
- **The exam printout consists of pages numbered 1 through 10.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the ten numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

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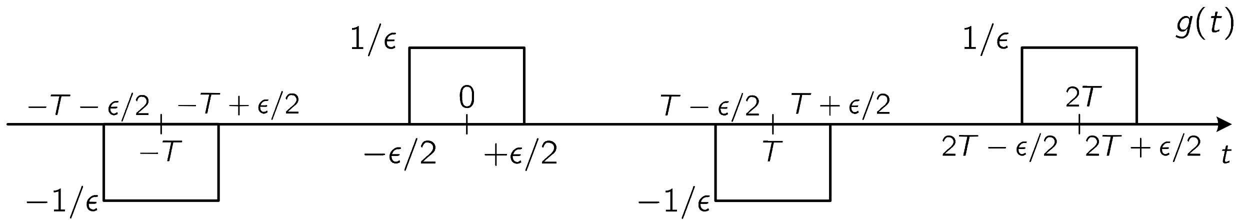
MT2.1 (70 Points) Consider the periodic continuous-time signal g shown in the figure below. The alternating square pulse pattern repeats outside the portion of the time axis depicted by the figure. And the parameter T is sufficiently large; in particular, assume $2\epsilon < T$.

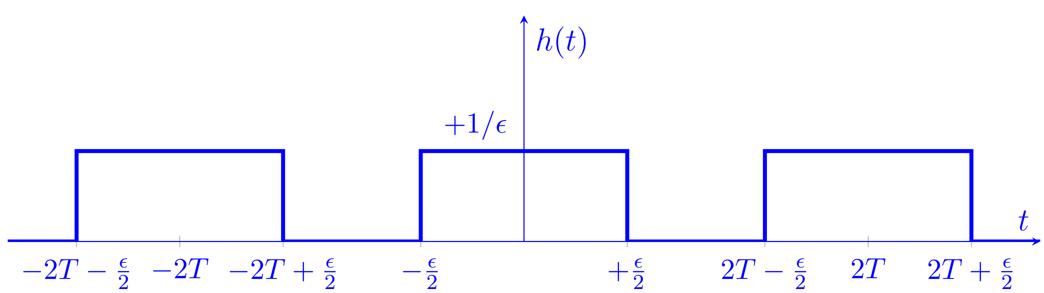


(a) (10 Points) Determine the fundamental period p and the fundamental frequency ω_0 of the signal g.

$$p = 2T \implies \omega_0 = \frac{2\pi}{2T} = \frac{\pi}{T} \tag{1}$$

(b) (20 Points) Determine a reasonably simple expression for the exponential Fourier series coefficients G_k (for all k in $\mathbb{Z}$) of the signal g.

Decompose g into $g(t) = h(t) - h(t-T)$, where $h(t)$ is shown below (not to scale).



$h(t)$ is periodic with fundamental period $2T$ and fundamental frequency $2\pi/(2T) = \pi/T$ (like g), and its Fourier series decomposition is:

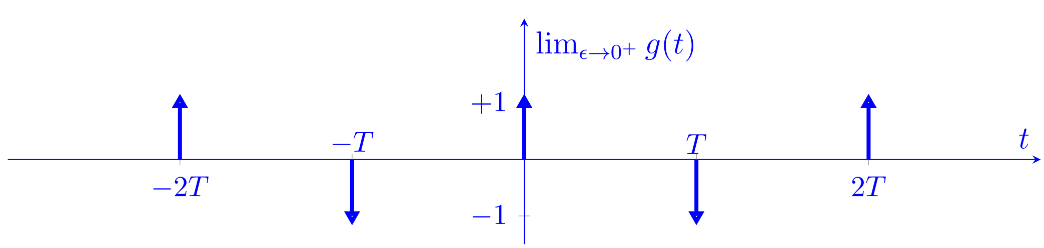
$$\begin{aligned}
H_k &= \frac{1}{2T}\int_{-\epsilon/2}^{\epsilon/2} \frac{1}{\epsilon} e^{-ik\omega_0 t} \mathrm{d}t && (2)\\
&= \frac{1}{2T\epsilon}\frac{1}{(-ik\omega_0)} e^{-ik\omega_0 t}\Big|_{t=-\epsilon/2}^{t=\epsilon/2} && (3)\\
&= \frac{1}{2\pi\epsilon(-ik)}\left(e^{-ik\omega_0(\epsilon/2)} - e^{ik\omega_0\epsilon/2}\right) && (4)\\
&= \frac{1}{k\pi\epsilon}\sin\left(k\omega_0\frac{\epsilon}{2}\right) && (5)
\end{aligned}$$

Because $h(t)$ and $g(t)$ have the same fundamental period, we can manipulate the synthesis equation form of $h(t)$ into a similar form for $g(t)$.

$$
\begin{aligned}
h(t) &= \sum_{k=-\infty}^{\infty} H_k e^{ik\omega_0 t} && (6)\\
\omega_0 T &= \frac{\pi}{T} T = \pi && (7)\\
g(t) &= h(t) - h(t-T) && (8)\\
&= \sum_{k=-\infty}^{\infty} H_k e^{ik\omega_0 t} - \sum_{k=-\infty}^{\infty} H_k e^{ik\omega_0 (t-T)} && (9)\\
&= \left(1 - e^{-ik\omega_0 T}\right) \sum_{k=-\infty}^{\infty} H_k e^{ik\omega_0 t} && (10)\\
&= \sum_{k=-\infty}^{\infty} \underbrace{H_k \left(1 - (-1)^k\right)}_{G_k} e^{ik\omega_0 t} && (11)\\
G_k &= \begin{cases} 0 & \text{if } k \bmod 2 = 0 \\ \frac{2}{k\pi\epsilon} \sin\left(k\omega_0 \frac{\epsilon}{2}\right) & \text{if } k \bmod 2 = 1 \end{cases} && (12)
\end{aligned}
$$

(c) (20 Points) Suppose $\epsilon \to 0$ in the figure above. Provide a well-labeled plot of the signal g in this limiting case and determine the corresponding exponential Fourier series coefficients G_k for g. You should be able to tackle this part even if you did not get through, or are not confident in your answer to, part (b).

The limiting case corresponds to each box morphing to a Dirac delta of strength $+1$ if the box is upward, and -1 if the box is downward. This is because each box has a signed area equal to $+1$ or -1, respectively.



To find G_k, choose a sufficiently small $\Delta > 0$ such that $\langle 2T \rangle = (-\Delta, 2T - \Delta)$. The exact value does not matter, so long as $\Delta \notin \{\ldots, -T, 0, T, \ldots\}$ (that is, a delta does not lie exactly on the interval boundary—the integral would not be well-defined). Using the sifting property, we get:

$$G_k = \frac{1}{2T}\int_{-\Delta}^{2T-\Delta} \left(\delta(t) - \delta(t-T)\right) e^{-ik\omega_0 t}\mathrm{d}t \tag{13}$$

$$= \frac{1}{2T}\int_{-\Delta}^{2T-\Delta} \left(e^0\delta(t) - e^{-ik\omega_0 T}\delta(t-T)\right)\mathrm{d}t \tag{14}$$

$$= \frac{1}{2T}\left(1 - (-1)^k\right) \tag{15}$$

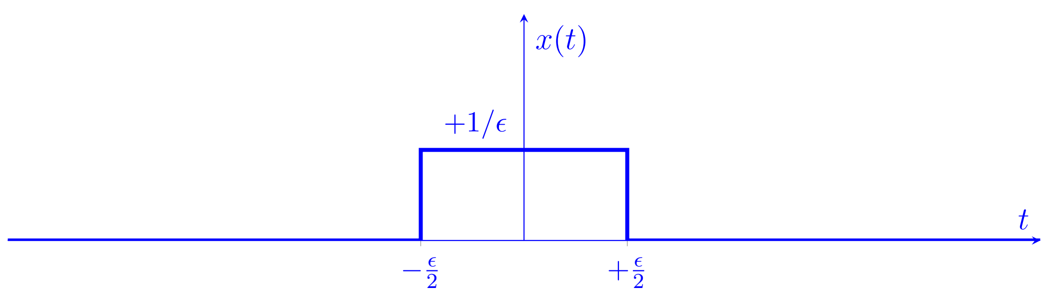
$$= \begin{cases} 0 & \text{if } k \bmod 2 = 0 \\ \frac{1}{T} & \text{if } k \bmod 2 = 1 \end{cases} \tag{16}$$

(d) (20 Points) Suppose g (with $\epsilon > 0$) is the impulse response of an LTI system whose input signal x is defined by the following:

$$\forall t, \quad x(t) = \begin{cases} g(t) & |t| < T/2 \\ 0 & \text{elsewhere.} \end{cases}$$

Provide a well-labeled plot of the output signal y corresponding to this input signal x. Remember that T is sufficiently large.

The input signal $x(t)$ just selects the $g(t)$ box centered at $t = 0$.



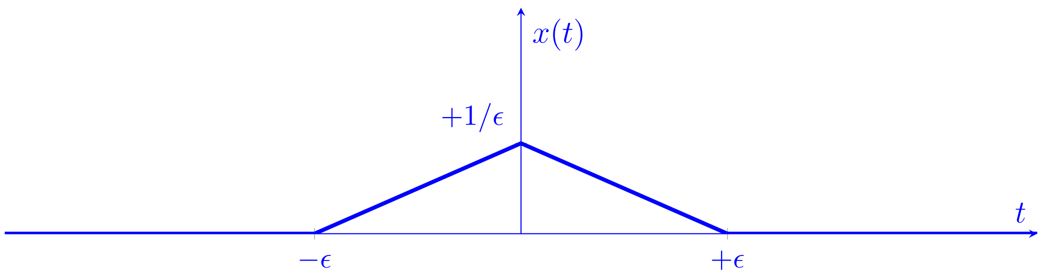
The output signal will be $y(t) = (g * x)(t)$. $g(t)$ can be written as a linear combination of $x(t)$:

$$g(t) = \sum_{k \text{ even}} x(t-kT) - \sum_{k \text{ odd}} x(t-kT) \tag{17}$$

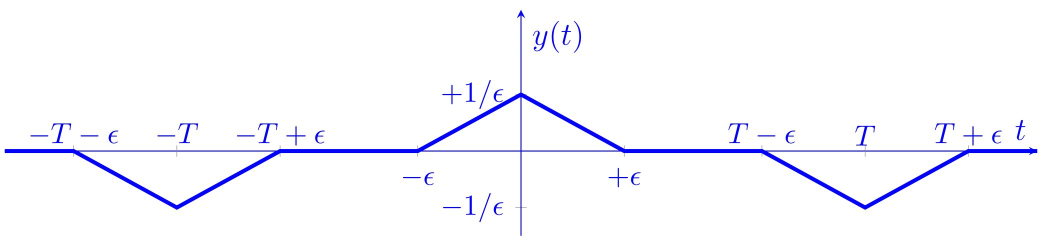
Since the convolution operator is distributive,

$$y(t) = \sum_{k \text{ even}} (x * x)(t-kT) - \sum_{k \text{ odd}} (x * x)(t-kT) \tag{18}$$

As it turns out, $(x*x)(t)$ produces a triangle of double that width (to see how, use the flip-and-shift method to find t for which the boxes in the convolution integrand overlap). The maximum value occurs when the boxes completely overlap.



Placing the copies of $(x*x)(t)$ in y, we have:



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MT2.2 (30 Points) DTFS ...

(a) (15 Points) Consider a positive integer p, and let $\omega_0 = 2\pi/p$. Prove that if $k \bmod p \neq 0$, then

$$\sum_{n=\langle p \rangle} \cos(k\omega_0 n) = \sum_{n=\langle p \rangle} \sin(k\omega_0 n) = 0.$$

Hint: Let p and ω_0 be the fundamental period and frequency, respectively, of a discrete-time signal x. Take advantage of what you know about the orthogonality of the basis functions Ψ_k in the complex exponential DTFS expansion

$$x(n) = \sum_{k=\langle p \rangle} X_k \underbrace{e^{ik\omega_0 n}}_{\Psi_k(n)}.$$

Let $\psi_k(n) = e^{ik\omega_0 n}$. We know that the following vectors are mutually orthogonal:

$$\boldsymbol{\Psi}_0 = \begin{bmatrix} \psi_0(0) \\ \psi_0(1) \\ \vdots \\ \psi_0(p-1) \end{bmatrix} \quad \boldsymbol{\Psi}_1 = \begin{bmatrix} \psi_1(0) \\ \psi_1(1) \\ \vdots \\ \psi_1(p-1) \end{bmatrix} \quad \cdots \quad \boldsymbol{\Psi}_{p-1} = \begin{bmatrix} \psi_{p-1}(0) \\ \psi_{p-1}(1) \\ \vdots \\ \psi_{p-1}(p-1) \end{bmatrix} \tag{19}$$

In particular,

$$\boldsymbol{\Psi}_0 \perp \boldsymbol{\Psi}_k, \forall k \in \{1, 2, \ldots, p-1\} \tag{20}$$

$$\implies \sum_{n=0}^{p-1} \underbrace{\psi_0(n)}_{1} \psi_k^\star(n) = \sum_{n=0}^{p-1} e^{-ik\omega_0 n} \tag{21}$$

$$= \sum_{n=0}^{p-1} \cos(k\omega_0 n) - i \sum_{n=0}^{p-1} \sin(k\omega_0 n) = 0 \tag{22}$$

$$\implies \sum_{n=\langle p \rangle} \cos(k\omega_0 n) = \sum_{n=\langle p \rangle} \sin(k\omega_0 n) = 0 \tag{23}$$

(b) (15 Points) Determine the smallest positive integers P_1 and P_2 such that each of the following statements is true; explain your reasoning even if you believe no finite integer value can be found.

$$\sum_{n=\langle P_1 \rangle} \cos\left(\frac{5\pi n}{3}\right) = 0 \qquad\qquad \sum_{n=\langle P_2 \rangle} e^{in} = 0.$$

Note: $\langle P_1 \rangle$ and $\langle P_2 \rangle$ refer to sets of <u>contiguous</u> integers.

$\cos(5\pi n/3)$ is periodic with fundamental period $P_1 = 6$. To see this, note that

$$\cos\left(\frac{5\pi}{3}(n+P_1)\right) = \cos\left(\frac{5\pi}{3}n + \frac{5\pi}{3}P_1\right) = \cos\left(\frac{5\pi}{3}n\right) \tag{24}$$

$$\implies \frac{5\pi}{3}P_1 = 2\pi k, k \in \mathbb{Z}_{>0} \implies P_1 = \frac{6}{5}k \tag{25}$$

$k = 5$ gives the smallest positive period P_1, which must be an integer. Therefore, $P_1 = 6$.

e^{in} is not periodic. To see why, suppose there is an integer p such that

$$e^{i(n+p)} = e^{in} \tag{26}$$

$$\implies \cos(n+p) = \cos(n) \text{ and } \sin(n+p) = \sin(p) \tag{27}$$

$$\implies \exists k \in \mathbb{Z}_{>0}, p = 2\pi k \tag{28}$$

However, this is a contradiction, since p/k is rational, and 2π is irrational. Therefore,

$$\sum_{n=\langle P_2 \rangle} e^{in} \neq 0 \text{ for any finite } P_2 \tag{29}$$

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MT2.3 (30 Points) The input x and corresponding output y of a continuous-time system G are

$$\forall t \in \mathbb{R}, \quad x(t) = \sum_{\ell=-\infty}^{+\infty} \delta(t-\ell) \quad \text{and} \quad y(t) = \sum_{m=-\infty}^{+\infty} (-1)^m \delta(t-m).$$

(a) (10 Points) Select the strongest assertion, which is true, from the choices below (and fill in the blank, as appropriate). Explain your choice succinctly, but clearly and convincingly.

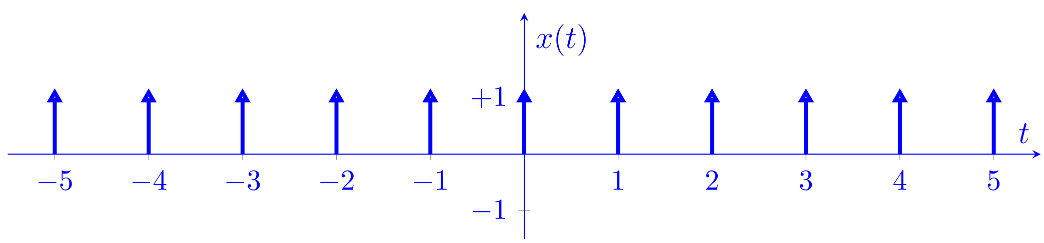
(i) G must be an LTI system, and its frequency response is ______.

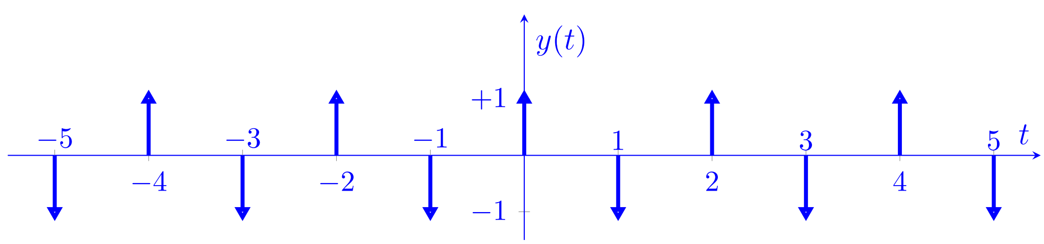
(ii) G can be an LTI system, and if it is, its frequency response is ______.

(iii) G cannot be an LTI system.

G cannot be LTI.

Method 1:





$p_x = 1 \implies \omega_{0,x} = 2\pi$, but $p_y = 2 \implies \omega_{0,y} = 2\pi/2 = \pi$, meaning new frequencies have been created.

Method 2: If the input to an LTI system has period p_x, the output is also periodic and has period at most $p_y = p_x$. Note that if y is the response to x, then $\widehat{x}(t) = x(t+p_x)$ produces $\widehat{y}(t) = y(t+t_p)$, but $\widehat{x}(t) = x(t) \implies \widehat{y}(t) = y(t)$.

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(b) (20 Points) Determine the complex exponential Fourier series expansion of the output y.

$$\omega_{0,y} = \pi \tag{30}$$

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{ik\omega_{0,y}t} \tag{31}$$

$$= \sum_{k=-\infty}^{\infty} Y_k e^{ik\pi t} \tag{32}$$

$$Y_k = \frac{1}{2}\int_{0^-}^{2^-} y(t)\, e^{-ik\pi t}\mathrm{d}t \tag{33}$$

$$= \frac{1}{2}\int_{0^-}^{2^-} \left(e^0\, \delta(t) - e^{-ik\pi(1)}\delta(t-1)\right) \mathrm{d}t \tag{34}$$

$$= \frac{1}{2}\left(1 - (-1)^k\right) \tag{35}$$

$$= \begin{cases} 0 & \text{if } k \bmod 2 = 0 \\ 1 & \text{if } k \bmod 2 = 1 \end{cases} \tag{36}$$

$$y(t) = \sum_{k \text{ odd}} e^{ik\pi t} \tag{37}$$

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MT2.4 (40 Points) Consider a discrete-time causal LTI filter H whose input x and output y satisfy the linear, constant-coefficient difference equation

$$y(n) = \alpha\, y(n-4) + x(n) - x(n-4), \tag{38}$$

where the parameter $\alpha = (0.95)^4$.

(a) (20 Points) Determine a reasonably simple expression for $H(\omega)$, the frequency response of the system, and provide a well-labeled plot of the magnitude response $|H(\omega)|$. You must explain any reasonable approximation you make to plot the magnitude response.

Take the DTFT of each side, use the time-shift property, then factor:

$$Y(\omega) = \alpha e^{-i4\omega} Y(\omega) + X(\omega) - e^{-i4\omega} X(\omega) \tag{39}$$

$$\left(1 - \alpha e^{-i4\omega}\right) Y(\omega) = \left(1 - e^{-i4\omega}\right) X(\omega) \tag{40}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 - e^{-i4\omega}}{1 - \alpha e^{-i4\omega}} \tag{41}$$

$$= \frac{e^{i4\omega} - 1}{e^{i4\omega} - (0.95)^4} \tag{42}$$

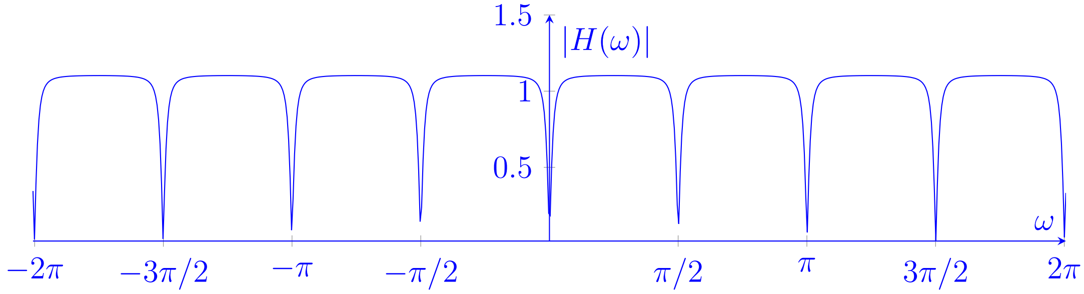
$$= \frac{\left(e^{i2\omega} - 1\right)\left(e^{i2\omega} + 1\right)}{\left(e^{i2\omega} - 0.95^2\right)\left(e^{i2\omega} + 0.95^2\right)} \tag{43}$$

$$= \frac{e^{i\omega} - 1}{e^{i\omega} - 0.95} \frac{e^{i\omega} + 1}{e^{i\omega} + 0.95} \frac{e^{i\omega} - i}{e^{i\omega} - 0.95i} \frac{e^{i\omega} + i}{e^{i\omega} + 0.95i} \tag{44}$$

- $\left(e^{i\omega} - 1\right) / \left(e^{i\omega} - 0.95\right) \approx 1$, except near $\omega = 0$, where this factor goes to zero.
- $\left(e^{i\omega} + 1\right) / \left(e^{i\omega} + 0.95\right) \approx 1$, except near $\omega = \pi$, where this factor goes to zero.
- $\left(e^{i\omega} - i\right) / \left(e^{i\omega} - 0.95i\right) \approx 1$, except near $\omega = \pi/2$, where this factor goes to zero.
- $\left(e^{i\omega} + i\right) / \left(e^{i\omega} + 0.95i\right) \approx 1$, except near $\omega = -\pi/2$, where this factor goes to zero.

Since only one of the factors needs to be zero for $H(\omega)$ to be zero, $H(\omega) = 0$ at $\omega \in \{\ldots, -\pi/2, 0, \pi/2, \pi, \ldots\}$.

This is a notch filter:



(b) (20 Points) The fundamental period of an input to the system is $p = 8$. If the DTFS expansion of the input signal is given by $x(n) = \sum_{k=0}^{p-1} X_k \, e^{ik\omega_0 n}$, determine the fundamental frequency ω_0 and a reasonable expression for the DTFS expansion of the output y.

$$x(n+8) = x(n), \forall n \in \mathbb{Z} \implies \omega_0 = \frac{2\pi}{p} = \frac{\pi}{4} \text{ rad/sample} \tag{45}$$

$$x(n) = \sum_{k=\langle 8 \rangle} X_k e^{ik(\pi/4)n} = X_0 + X_1 e^{i(\pi/4)n} + X_2 e^{i(\pi/2)n} + \cdots + X_7 e^{i(7\pi/4)n} \tag{46}$$

The system removes all even multiples of frequency $\pi/4$ (that is, the frequencies at $\omega \in \{\ldots, -\pi, -\pi/2, 0, \pi/2, \pi, \ldots\}$), and passes the rest (essentially) unchanged.

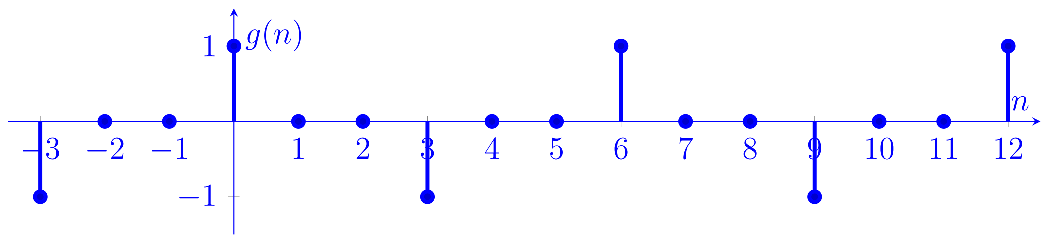
$$y(n) = X_1 e^{i(\pi/4)n} + X_3 e^{i(3\pi/4)n} + X_5 e^{i(5\pi/4)n} + X_7 e^{i(7\pi/4)n} \tag{47}$$

MT2.5 (20 Points) Consider a periodic discrete-time signal g described as follows:

$$\forall n, \quad g(n) = \begin{cases} +1 & n = 0, \pm 6, \pm 12, \ldots \\ -1 & n = \pm 3, \pm 9, \ldots \\ 0 & \text{elsewhere.} \end{cases}$$

Determine the discrete-time Fourier series coefficients G_k for this signal.

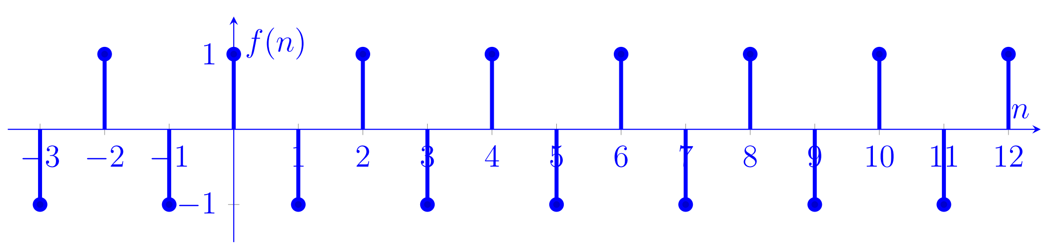
The plot of g is



g has fundamental period $p_g = 6$ and fundamental frequency $\omega_0 = 2\pi/p_g = \pi/3$. From the DTFS analysis equation, the coefficients are:

$$\begin{aligned} G_k &= \frac{1}{p} \sum_{n=\langle p \rangle} g(n)\, e^{-ik\omega_0 n} && (48) \\ &= \frac{1}{6} \sum_{n=0}^{5} g(n) e^{-ik(\pi/3)n} && (49) \\ &= \frac{1}{6} \left(g(0) + g(3)\, e^{-ik(\pi/3)3} \right) && (50) \\ &= \frac{1}{6} \left(1 - (-1)^k \right) && (51) \\ G_0 &= G_2 = G_4 = 0 && (52) \\ G_1 &= G_3 = G_5 = \frac{1}{6}(2) = \frac{1}{3} && (53) \end{aligned}$$

Alternatively, we see that g is an upsampled version of the signal f, which is defined as follows:



You can show that the DTFS decomposition of f, which has fundamental period $p_f = 2$, is $F_0 = 0$, $F_1 = 1$. The upsampling is by a factor of $N = 3$, so the coefficients of f and g are related by:

$$G_k = \frac{1}{N} F_{k \bmod p_f} = \frac{1}{3} F_{k \bmod 2} \tag{54}$$

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FORMULAS & TABLES

Discrete-Time Fourier Series (DTFS) Complex exponential Fourier series synthesis and analysis equations for a periodic discrete-time signal having period p:

$$x(n) = \sum_{k=\langle p \rangle} X_k \, e^{ik\omega_0 n} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \sum_{n=\langle p \rangle} x(n) \, e^{-ik\omega_0 n} \; ,$$

where $p = \dfrac{2\pi}{\omega_0}$ and $\langle p \rangle$ denotes a suitable discrete interval of length p (i.e., an interval containing p contiguous integers). For example, $\displaystyle\sum_{k=\langle p \rangle}$ may denote $\displaystyle\sum_{k=0}^{p-1}$ or $\displaystyle\sum_{k=1}^{p}$.

Continuous-Time Fourier Series (CTFS) Complex exponential Fourier series synthesis and analysis equations for a periodic continuous-time signal having period p:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{ik\omega_0 t} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) \, e^{-ik\omega_0 t} dt \; ,$$

where $p = \dfrac{2\pi}{\omega_0}$ and $\langle p \rangle$ denotes a suitable continuous interval of length p. For example, $\displaystyle\int_{\langle p \rangle}$ can denote $\displaystyle\int_0^p$.