# MATH 121B. MIDTERM 1. 2020 SPRING

Total 50 points

(1) Are they vector spaces? Just true or false. (5 points)
  (a) The points in $\mathbb{R}^2$ satisfying $x + y = 1$.
  (b) The points in $\mathbb{R}^3$ satisfying $x + y + z = 0$.
  (c) The points in $\mathbb{R}^3$ satisfying $x^2 + y^2 + z^2 = 0$.
  (d) The points in $\mathbb{R}^3$ satisfying $xyz = 0$.
  (e) The set of polynomials $f(t)$ such that degree of $f(t)$ is at most $5$, and satisfies $f(1) = 0$.

(2) Let $V$ be the vector space of smooth functions on the interval $[0, 1]$. Are the following $\phi : V \to \mathbb{R}$ linear functions on $V$? Just true or false. (5 points)
  (a) $\phi(f) = f(1/2)$, that is, $\phi$ sends an element $f(t) \in V$ to its value at $t = 1/2$.
  (b) $\phi(f) = \int_0^1 f(t)dt$
  (c) $\phi(f) = \int_0^1 f(t)^2 dt$
  (d) $\phi(f) = f'(1/3) + f''(2/3)$
  (e) $\phi(f) = f(1/3) \cdot f(2/3)$

(3) Let $V$ be a vector space of dimension $3$ with basis $e_1, \cdots, e_3$. Let $\widetilde{e}_1 = e_1, \widetilde{e}_2 = e_1 + e_2, \widetilde{e}_3 = e_1 + e_2 + e_3$ be a new basis of $V$. (10 points)
  (a) What is the dimension of $V \otimes V$? (2 pts)
  (b) Write down a basis of $V \wedge V$ using $e_i$s. (2 pts)
  (c) Suppose we have vector $\mathbf{v} = 3e_2 + 5e_3$, can you express $\mathbf{v}$ in the basis $\widetilde{e}_i$? (3 pts)
  (d) Suppose we have a tensor $T = e_2 \otimes e_3$, can you express $T$ in the basis $\widetilde{e}_i$? (3 pts)

(4) Let $V = \mathbb{R}^2$ with coordinates $(x, y)$ and with $g = dx^2 + dy^2$. Introduce a new basis $\mathbf{e}_1 = (-1, 0)$ and $\mathbf{e}_2 = (-1, 1)$ of $V$. Introduce new linear coordinates $(u, v)$ on $\mathbb{R}^2$, such that $u(\mathbf{e}_1) = 1, u(\mathbf{e}_2) = 0$ and $v(\mathbf{e}_1) = 0, v(\mathbf{e}_2) = 1$. (10 points)
  (a) Let $\mathbf{v} = (1, 1)$ (in the $(x, y)$ coordinate system). Write $\mathbf{v}$ as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$. (5 points)
  (b) What is the metric tensor $g$ (or $ds^2$ in Boas term), expressed using $du$ and $dv$? (5 points)

(5) Let $(u, v)$ be a new set of coordinates near $(0, 0)$ on $\mathbb{R}^2$, which is related to the Cartesian coordinate as the following. (20 points)

$$x = u + (u^2 - v^2)/2, \quad y = v + uv$$

(where we require $u > -1/2$. )
  (a) what is $ds^2$ in $du, dv$? (10 points)
  (b) Let $f = u^2 - v^2$, what is the gradient of $f$? (5 points)
  (c) Let $V = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}$ be a vector field, (in Boas notation, $V = u\mathbf{a}_v - v\mathbf{a}_u$). What is the divergence of $V$? (5)

## 1. Solution

(1) Are they vector spaces? Just true or false. (5 points)
  (a) The points in $\mathbb{R}^2$ satisfying $x + y = 1$. False
  (b) The points in $\mathbb{R}^3$ satisfying $x + y + z = 0$. True
  (c) The points in $\mathbb{R}^3$ satisfying $x^2 + y^2 + z^2 = 0$. True or False both OK. If we had been working over , then it is False. Since we are working over $\mathbb{R}$, the actually solution is just the origin, hence it is a trivial linear space.
  (d) The points in $\mathbb{R}^3$ satisfying $xyz = 0$. False.
  (e) The set of polynomials $f(t)$ such that degree of $f(t)$ is at most 5, and satisfies $f(1) = 0$. True.

(2) Let $V$ be the vector space of smooth functions on the interval $[0, 1]$. Are the following $\phi : V \to \mathbb{R}$ linear functions on $V$? Just true or false. (5 points)
  (a) $\phi(f) = f(1/2)$, that is, $\phi$ sends an element $f(t) \in V$ to its value at $t = 1/2$. True
  (b) $\phi(f) = \int_0^1 f(t)dt$ True
  (c) $\phi(f) = \int_0^1 f(t)^2 dt$ False
  (d) $\phi(f) = f'(1/3) + f''(2/3)$ True
  (e) $\phi(f) = f(1/3) \cdot f(2/3)$ False

(3) Let $V$ be a vector space of dimension 3 with basis $e_1, \cdots, e_3$. Let $\widetilde{e}_1 = e_1, \widetilde{e}_2 = e_1 + e_2, \widetilde{e}_3 = e_1 + e_2 + e_3$ be a new basis of $V$. (10 points)
  (a) What is the dimension of $V \otimes V$? (2 pts) 9
  (b) Write down a basis of $V \wedge V$ using $e_i$s. (2 pts) $e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3$
  (c) Suppose we have vector $\mathbf{v} = 3e_2 + 5e_3$, can you express $\mathbf{v}$ in the basis $\widetilde{e}_i$? (3 pts)
  (d) Suppose we have a tensor $T = e_2 \otimes e_3$, can you express $T$ in the basis $\widetilde{e}_i$? (3 pts)

  For the last two problem, we can plug in $e_2 = \widetilde{e}_2 - \widetilde{e}_1$ and $e_3 = \widetilde{e}_3 - \widetilde{e}_2$, to get

$$\mathbf{v} = 3(\widetilde{e}_2 - \widetilde{e}_1) + 5(\widetilde{e}_3 - \widetilde{e}_2)$$

  and

$$T = e_2 \otimes e_3 = (\widetilde{e}_2 - \widetilde{e}_1) \otimes (\widetilde{e}_3 - \widetilde{e}_2).$$

  Then, one may open the parenthesis and expand if one want.

(4) Let $V = \mathbb{R}^2$ with coordinates $(x, y)$ and with $g = dx^2 + dy^2$. Introduce a new basis $\mathbf{e}_1 = (-1, 0)$ and $\mathbf{e}_2 = (-1, 1)$ of $V$. Introduce new linear coordinates $(u, v)$ on $\mathbb{R}^2$, such that $u(\mathbf{e}_1) = 1, u(\mathbf{e}_2) = 0$ and $v(\mathbf{e}_1) = 0, v(\mathbf{e}_2) = 1$. (10 points)
  (a) Let $\mathbf{v} = (1, 1)$ (in the $(x, y)$ coordinate system). Write $\mathbf{v}$ as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$. (5 points)
  $\mathbf{v} = e_2 - 2e_1$
  (b) What is the metric tensor $g$ (or $ds^2$ in Boas term), expressed using $du$ and $dv$? (5 points) $u_1 = u, u_2 = v$ are dual basis of $e_1, e_2$, hence the coefficients in front of $du_i \otimes du_j$ is $g(e_i, e_j)$, we get

$$g = g(e_1, e_1)du \otimes +g(e_1, e_2)du \otimes dv + g(e_2, e_1)dv \otimes du + g(e_2, e_2)dv \otimes dv$$
$$= du \otimes du + du \otimes dv + dv \otimes du + 2dv \otimes dv.$$

(5) Let $(u, v)$ be a new set of coordinates near $(0, 0)$ on $\mathbb{R}^2$, which is related to the Cartesian coordinate as the following. (20 points)

$$x = u + (u^2 - v^2)/2, \quad y = v + uv$$

(where we require $u > -1/2$. )

(a) what is $ds^2$ in $du, dv$? (10 points)
(b) Let $f = u^2 - v^2$, what is the gradient of $f$? (5 points)
(c) Let $V = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}$ be a vector field, (in Boas notation, $V = u\mathbf{a}_v - v\mathbf{a}_u$). What is the divergence of $V$? (5)

We plug in $dx = (1 + u)du - vdv$ and $dy = (1 + u)dv + vdu$ into $ds^2 = dx^2 + dy^2$, we get

$$ds^2 = [(1 + u)^2 + v^2](du^2 + dv^2]$$

It is a orthogonal coordinate.

(2) $\nabla f = g^{uu}\partial_u f \partial_u + g^{vv}\partial_v f \partial_v = \frac{2u\partial_u - 2v\partial_v}{(1+u)^2+v^2}$

(3) $\nabla \cdot V = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}V^i) = \frac{1}{(1+u)^2+v^2}(\partial_u((-(1 + u)^2 + v^2)v) + \partial_v((1 + u)^2 + v^2)u) =$