Math 121A Midterm 2 Nov 7, 2019

Name: Solution.

- You have 80 minutes to complete the exam.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- **The Laplace Transform table is provided in a separated sheet of paper.**

Good Luck!

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total	100	

1. (20 points) Use contour integral to compute the following integral.

(1) (5 pts)

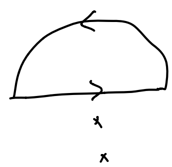
$$\int_0^{2\pi} (1+\cos\theta + i\sin\theta)^2 d\theta$$

$$\int_0^{2\pi} (1+e^{i\theta})^2 d\theta = \oint_{|z|=1} (1+z)^2 \frac{dz}{iz}$$

$$= \frac{2\pi i}{i} \operatorname{Res}_{z=0} \left(\frac{(1+z)^2}{z} \right) = 2\pi$$

(2) (3 pts)

$$\int_{-\infty}^{+\infty} \frac{1}{(x+i)(x+2i)} dx$$



we complete the contour using upper semi-circle it encloses no pole. Hence the integral $= 0$.

(3) (7 pts)

$$\int_{-\infty}^{+\infty} \frac{\cos x}{(x+i)(x+2i)} dx$$

$$= \int_{-\infty}^{+\infty} \frac{\frac{1}{2}(e^{ix}+e^{-ix})}{(x+i)(x+2i)} dx = \frac{1}{2}(I_1 + I_2).$$

$$I_1 = \int_{-\infty}^{+\infty} \frac{e^{ix}}{(x+i)(x+2i)} dx = \oint \frac{e^{ix}}{(x+i)(x+2i)} dx = 0$$

$$I_2 = \int_{-\infty}^{+\infty} \frac{e^{-ix}}{(x+i)(x+2i)} dx = \oint \frac{e^{-ix}}{(x+i)(x+2i)} dx = (-2\pi i)\left(\frac{e^{-1}}{i} + \frac{e^{-2}}{-i} \right)$$

(4) (5 pts)

$$P.V. \int_{-\infty}^{+\infty} \frac{1}{(x+1)(x+2)} dx$$

$$= 2\pi i \cdot \left(\frac{1}{2} \operatorname{Res}(x=-1) + \frac{1}{2} \operatorname{Res}(x=-2) \right)$$

$$= \frac{2\pi i}{2} \cdot \left(\frac{1}{-1+2} + \frac{1}{-2+1} \right)$$

$$= 0$$

Remark: most common mistake about (3) is say

$$\text{integral} = \text{Re} \int_{-\infty}^{+\infty} \frac{e^{iz}}{(z+i)(z+2i)} \, dz$$

However, the original integral is.

$$\int_{-\infty}^{+\infty} \frac{\text{Re}(e^{iz})}{(z+i)(z+2i)} \, dz.$$

Because $(z+i)(z+2i)$ is not a real number,

$$\frac{\text{Re}(e^{iz})}{(z+i)(z+2i)} \neq \text{Re}\left(\frac{e^{iz}}{(z+i)(z+2i)}\right)$$

2. (10 points) Write down the general solutions for the following equations

(1) (5pts)

$$y' + 3y = -1$$

(1) $y_p = -\frac{1}{3}$ particular sol'n

(2) $y_c = c \cdot e^{-3x}$

$y = y_p + y_c = -\frac{1}{3} + ce^{-3x}$

(2) (5pts)

$$y' + (x-1)y = 0$$

$y = e^{-\int (x-1)\,dx}$

$= ce^{-\frac{x^2}{2} + x}$

3. (10 points) Write down the general solutions to the following equations

(1) (3 pts) $y'' + 4y' + 4y = 0$

$$(D^2+4D+4)\,y=0$$

$$(D+2)^2\;y=0$$

$$y = c_1\cdot e^{-2x} + c_2\cdot x\cdot e^{-2x}$$

(2) (7 pts) $y'' + 4y = e^x$

First, we guess a particular solution

$$y_p = a\cdot e^x.$$

plug ~~it~~ in, we have

$$a\cdot e^x + 4\cdot a\cdot e^x = e^x$$

$$\Longleftrightarrow \quad 5a = 1$$

$$a = 1/5$$

Then, we solve for y_c.

$$y'' + 4y = 0$$

$$\Longleftrightarrow \quad (D^2+4)y = 0$$

$$\Longleftrightarrow \quad (D+2i)(D-2i)\,y = 0.$$

$$y_c = c_1 e^{2ix} + c_2\cdot e^{-2ix}.$$

Thus $\quad y = \frac{1}{5}e^x + c_1 e^{2ix} + c_2\cdot e^{-2ix}$

4. (10 points) Answer the following questions about δ functions

(1) (2pts) $\int_{-\infty}^{\infty} \delta(x) \cos x dx = ?$

$$= \cos(0) = 1$$

(2) (2pts) $\int_{-\infty}^{\infty} [\delta(x) + \delta(2x-2)] \sin x dx = ?$

$$= \sin(0) + \frac{1}{2} \sin(1)$$

$$= \frac{1}{2} \sin(1).$$

(3) (2pts) $\int_{-\infty}^{\infty} \delta'(x+1) e^{2x} dx = ?$

$$= -\left(e^{2x}\right)'\Big|_{x=-1}$$

$$= -2e^{-2}$$

(4) (4 pts) Solve for $y(x)$ that satisfies the following condition

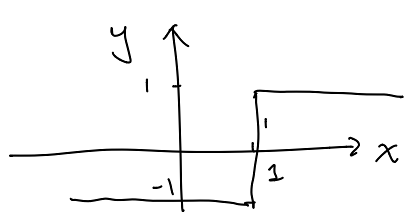
$$\begin{cases} y' = 2\delta(x-1) \\ y(0) = -1 \end{cases}$$

Draw the graph of $y(x)$.

$$y(x) = C + 2 H (x-1)$$

use $y(0) = -1$, we get $C = -1$.

Thus $y = -1 + H(x-1)$



5. (10 points)Laplace transformation. You can either use the Laplace transformation table, or the following integral to find the Laplace transformation.

$$F(p) = \int_0^\infty f(t)e^{-pt}dt.$$

(1) (2pts) $f(t) = 1 + t$

$$F(p) = \frac{1}{p} + \frac{1}{p^2}$$

(2) (2pts) $f(t) = e^t$

$$F(p) = \frac{1}{p-1}$$

(3) (2pts) $f(t) = \sin(2t)$

$$F(p) = \frac{2}{p^2+2^2}$$

(4) (4pts) $f(t) = \int_0^t \sin(t-\tau)\tau d\tau$ (Hint: use the convolution for Laplace transform)

$f(t)$ is the convolution of $\sin t$ and t.

$$\sin t \longrightarrow \frac{1}{p^2+1}$$

$$t \longrightarrow \frac{1}{p^2}$$

$$\therefore f(t) \rightarrow \frac{1}{p^2} \cdot \frac{1}{p^2+1}$$

6. (10 points) Find the inverse Laplace transform of the following function. You can either use the Laplace transformation table, or use the inverse Laplace transform integral

$$f(t) = \frac{1}{2\pi i} \int_{s-i\infty}^{s+i\infty} e^{pt} F(p) dp,$$

where s is a sufficiently large real number.

(1) (3pts)

$$F(p) = \frac{1}{p^2 + 1}$$

One can use the table L3. to get

$$f(t) = \sin(t).$$

Or, use integral to get $\text{Res}\left(\frac{e^{pt}}{p^2+1}\right)$

$$= \frac{e^{it}}{2i} + \frac{e^{-it}}{-2i} = \sin(t).$$

(2) (4pts) (Hint: try partial fraction)

$$F(p) = \frac{1}{(p+1)(p+2)(p+3)}$$

Use partial fraction, we get

$$F(p) = \frac{1}{p+1} \cdot \frac{1}{+2} + \frac{1}{p+2} \frac{1}{-1} + \frac{1}{p+3} \frac{1}{2}$$

Thus

$$f(t) = \frac{1}{2} e^{-t} - e^{-2t} + \frac{1}{2} e^{-3t}$$

(3) (3pts) Hint: write the numerator as $p = (p+2) - 2$

$$F(p) = \frac{p}{(p+2)^4}$$

$$F(p) = \frac{p+2-2}{(p+2)^4} = \frac{1}{(p+2)^3} - \frac{2}{(p+2)^4}$$

L6: $$f(t) = \frac{1}{2!} t^2 e^{-2t} - \frac{2}{3!} t^3 e^{-2t}$$

7. (10 points) Use Laplace transform to solve the differential equation

$$\begin{cases} y'' + y = e^{-3t} \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

You may use that

$$L.T.(y) = Y(p), \quad L.T.(y'') = p^2 Y(p) - py(0) - y'(0)$$

By Laplace transform, we have

$$L.T.(e^{-3t}) = \frac{1}{p+3}.$$

Then, we have

$$LT(y'') + LT(y) = LT(e^{-3t})$$

$$p^2 Y(p) - 1 + Y(p) = \frac{1}{p+3}$$

$$\therefore \quad (1+p^2) Y(p) = 1 + \frac{1}{p+3}.$$

$$Y(p) = \frac{1}{p^2+1} + \frac{1}{(p^2+1)(p+3)}$$

$$= \frac{1}{p^2+1} + \frac{1}{(p+i)(p-i)(p+3)}$$

Use partial fraction, we get

$$\frac{1}{(p+i)(p-i)(p+3)} = \frac{1}{p+i}\,\frac{1}{(-i-i)(-i+3)} + \frac{1}{p-i}\,\frac{1}{(i+i)(i+3)}$$
$$+ \frac{1}{p+3}\,\frac{1}{(-3+i)(-3-i)}$$

Hence

$$f(t) = \sin t + e^{-it}\frac{1}{(3-i)(-2i)} + e^{it}\frac{1}{(3+i)(2i)}$$
$$+ e^{3t}\frac{1}{10}$$

8. (10 points) Solve the following equation with $a > 0$ and $x > 0$. [1]

$$\begin{cases} y''(x) + y'(x) = \delta(x-a) \\ y(0) = 0, y'(0) = 0 \end{cases}$$

We may using matching coeff method.

For $x < a$, $y_- = 0$ is the solution

For $x > a$, the general solution to

$$y'' + y' = 0 \iff D(D+1)y = 0$$

is $y_+(x) = C_1 + C_2 \cdot e^{-x}$.

At $x = a$, we need to have

$$y_-(a) = y_+(a).$$

and $y_+'(a) - y_-'(a) = 1$

Hence

$$\begin{cases} 0 = C_1 + C_2 e^{-a} \\ -C_2 \cdot e^{-a} = 1 \end{cases}$$

$\therefore$

$$\begin{cases} C_2 = -e^{a} \\ C_1 = 1. \end{cases}$$

$\therefore$

$$y = \begin{cases} 0 & x < a \\ 1 - e^{-(x-a)} & x > a \end{cases}$$

[1]The solution is the Green function $G(x; a)$ for this problem.

#8. Alternatively, we may use Laplace transform:

$$(p^2+p)\,Y(p) = L(\delta(x-a))$$

$$Y(p) = \frac{1}{p^2+p} \cdot L(\delta(x-a))$$

The inverse Laplace transform of $\frac{1}{p^2+p}$ is

$$\frac{1}{p^2+p} = \frac{1}{p(p+1)} = \frac{1}{p} - \frac{1}{p+1} \rightsquigarrow 1-e^{-x}$$

Let $F(p) = \frac{1}{p^2+p}$, $G(p) = L(\delta(x-a))$

and $f(x) = 1-e^{-x}$, $g(x) = \delta(x-a)$

be the original function. then

$$y(x) = (f*g)(x) = \int_0^x f(x-\tau)\,g(\tau)\,d\tau$$

$$= \int_0^x f(x-\tau)\,\delta(\tau-a)\,d\tau$$

$$= \begin{cases} 1-e^{-(x-a)} & a < x \\ 0 & a > x \end{cases}$$

9. (10 points) The Green function $G(x; x_0)$ for the following problem

$$\begin{cases} \frac{d^2}{dx^2} G(x; x_0) = \delta(x - x_0), & 0 < x, x_0 < 1 \\ G(0; x_0) = G(1; x_0) = 0 \end{cases}$$

is given by

$$G(x; x_0) = \begin{cases} x(x_0 - 1) & \text{if } x \le x_0 \\ x_0(x - 1) & \text{if } x_0 < x. \end{cases}$$

We are going to use the given Green function to solve the following equation

$$\begin{cases} y''(x) = f(x), & 0 < x < 1 \\ y(0) = y(1) = 0 \end{cases}$$

(1) (2 pts) Write down the general formula that expresses $y(x)$ using an integral involving G and f.

$$y(x) = \int_0^1 G(x; x_0) f(x_0) \, dx_0$$

(2) (3 pts) Use Green function to solve for y when $f(x) = 3\delta(x - 0.3)$.

$$y(x) = 3 \cdot G(x; 0.3)$$

$$= \begin{cases} 3 \cdot x \cdot (-0.7) & x < 0.3 \\ 3 \cdot 0.3 (x - 1) & x > 0.3 \end{cases}$$

(3) (5pts) Use Green function to solve for y when $f(x) = x$.

$$y(x) = \int_0^1 G(x; x_0)\, f(x_0)\, dx_0$$

$$= \int_0^x G(x; x_0)\, f(x_0)\, dx_0$$

$$+ \int_x^1 G(x; x_0)\, f(x_0)\, dx_0$$

$$= \int_0^x x_0 (x-1) \cdot x_0\, dx_0$$

$$+ \int_x^1 x(x_0 - 1)\, x_0\, dx_0$$

$$= (x-1) \cdot \frac{1}{3} x_0^3 \Big|_{x_0=0}^{x_0=x} + x \cdot \left(\frac{x_0^3}{3} - \frac{x_0^2}{2} \right) \Big|_{x_0=x}^{x_0=1}$$

$$= (x-1) \cdot \frac{1}{3} \cdot x^3 + x \cdot \left[\left(\frac{1}{3} - \frac{1}{2} \right) - \left(\frac{x^3}{3} - \frac{x^2}{2} \right) \right]$$

$$= \frac{1}{3} x^4 - \frac{1}{3} x^3 + \; - \frac{1}{6} x - \frac{x^4}{3} + \frac{x^3}{2}$$

$$= \frac{1}{6} x^3 - \frac{1}{6} x.$$

Check: $y(0) = y(1) = 0$,

$y''(x) = x$. ✓.