Name: ____

- You have 80 minutes to complete the exam.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- Please write neatly. Answers which are illegible for the reader cannot be given credit.
- **The Laplace Transform table is provided in a separated sheet of paper.**

Good Luck!

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total	100	

1. (20 points) Use contour integral to compute the following integral.

(1) (5 pts)

$$\int_0^{2\pi} (1 + \cos\theta + i\sin\theta)^2 d\theta$$

(2) (3 pts)

$$\int_{-\infty}^{+\infty} \frac{1}{(x+i)(x+2i)} dx$$

(3) (7 pts)

$$\int_{-\infty}^{+\infty} \frac{\cos x}{(x+i)(x+2i)} dx$$

(4) (5 pts)

$$P.V. \int_{-\infty}^{+\infty} \frac{1}{(x+1)(x+2)} dx$$

2. (10 points) Write down the general solutions for the following equations

(1) (5pts)

$$y' + 3y = -1$$

(2) (5pts)

$$y' + (x-1)y = 0$$

3. (10 points) Write down the general solutions to the following equations

(1) (5 pts) $y'' + 4y' + 4y = 0$

(2) (5 pts) $y'' + 4y = e^x$

4. (10 points) Answer the following questions about δ functions

(1) (2 pts) $\int_{-\infty}^{\infty} \delta(x) \cos x dx = ?$

(2) (2 pts) $\int_{-\infty}^{\infty} [\delta(x) + \delta(2x - 2)] \sin x dx = ?$

(3) (3 pts) $\int_{-\infty}^{\infty} \delta'(x + 1) e^{2x} dx = ?$

(4) (3 pts) Solve for $y(x)$ that satisfies the following condition

$$\begin{cases} y' = 2\delta(x - 1) \\ y(0) = -1 \end{cases}$$

Draw the graph of $y(x)$.

5. (10 points)Laplace transformation. You can either use the Laplace transformation table, or the following integral to find the Laplace transformation.

$$F(p) = \int_0^\infty f(t)e^{-pt}dt.$$

(1) (2pts) $f(t) = 1 + t$

(2) (2pts) $f(t) = e^t$

(3) (2pts) $f(t) = \sin(2t)$

(4) (4pts) $f(t) = \int_0^t \sin(t-\tau)\tau d\tau$ (Hint: use the convolution for Laplace transform)

6. (10 points) Find the inverse Laplace transform of the following function. You can either use the Laplace transformation table, or use the inverse Laplace transform integral

$$f(t) = \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} e^{pt}F(p)dp,$$

where s is a sufficiently large real number.

(1) (3pts)

$$F(p) = \frac{1}{p^2+1}$$

(2) (4pts) (Hint: try partial fraction)

$$F(p) = \frac{1}{(p+1)(p+2)(p+3)}$$

(3) (3pts) Hint: write the numerator as $p = (p+2) - 2$

$$F(p) = \frac{p}{(p+2)^4}$$

7. (10 points) Use Laplace transform to solve the differential equation for $t > 0$,

$$\begin{cases} y''(t) + y(t) = e^{-3t} \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

(a) (5pt) Use the following to get an equation for $Y(p)$ and solve for $Y(p)$.

$$LT(y) = Y(p), \quad LT(y'') = p^2Y(p) - py(0) - y'(0), \quad LT(e^{-3t}) = \frac{1}{p+3}$$

(b) (5pt) Find $y(t)$ from $Y(p)$. (Hint: use partial fraction, or inverse Laplace transform)

8. (10 points) Solve the following equation with $a > 0$ and $x > 0$. [1]

$$\begin{cases} y''(x) + y'(x) = \delta(x-a) \\ y(0) = 0, y'(0) = 0 \end{cases}$$

Hint: One method to solve is the following. Let $y_-(x)$ be the solution for the region $0 < x < a$, and $y_+(x)$ be the solution for the region $x > a$. Write down their general solution with undetermined coefficients. Then determine the coefficients using the given boundary condition and the matching conditions:

$$y_-(a) = y_+(a), \quad y'_+(a) - y'_-(a) = 1.$$

[1] The solution is the Green function $G(x; a)$ for this problem.

9. (10 points) The Green function $G(x; x_0)$ for the following problem

$$\begin{cases} \frac{d^2}{dx^2} G(x; x_0) = \delta(x - x_0), & 0 < x, x_0 < 1 \\ G(0; x_0) = G(1; x_0) = 0 \end{cases}$$

is given by

$$G(x; x_0) = \begin{cases} x(x_0 - 1) & \text{if } x \leq x_0 \\ x_0(x - 1) & \text{if } x_0 < x. \end{cases}$$

We are going to use the given Green function to solve the following equation

$$\begin{cases} y''(x) = f(x), & 0 < x < 1 \\ y(0) = y(1) = 0 \end{cases}$$

(1) (2 pts) Write down the general formula that expresses $y(x)$ using an integral involving G and f.

(2) (3 pts) Use Green function to solve for y when $f(x) = 3\delta(x - 0.3)$.

(3) (5pts) Use Green function to solve for y when $f(x) = x$.