MATH 54 FINAL

December 19 2019 11:30-2:30pm

Your Name Student ID	SOLUTIONS

No material other than simple writing utensils may be used.

Do not turn this page until you are instructed to do so.

In the event of an emergency or fire alarm leave your exam at your seat and meet with your GSI or professor outside.

This exam consists of 5 problems, each has questions (a), (b), (c) that test skills at level C, B, A in the general topic areas
1) Matrix Algebra
2) Abstract Linear Algebra
3) Ordinary Differential Equations
4) Linear Systems of Ordinary Differential Equations
5) Fourier Series and Partial Differential Equations

Each part of (a) yields full or no credit, and you don't need to show work. *To ensure credit please put each answer (and only the final answer) into the given box. Empty boxes will receive automatic score 0, so if your answer is elsewhere, put at least an arrow into the box.*

Parts (b),(c) can yield partial credit, in particular for explanations and documentation of your approach, even when you don't complete the calculation. In particular, if you recognize your result to be wrong (e.g. by checking!), stating this will yield extra credit. On the other hand, wrong or irrelevant statements mixed with correct work may result in reduced credit.

When asked to explain/show/prove, you should make clear and unambiguous statements that would be accessible to another student. In particular, use words or arrows to indicate how formulas relate to each other. You may use any theorems or facts stated in the lecture notes, script, and the book sections covered by the course – after stating them clearly. If you wish to use theorems or facts that you may know from other sources, you need to include proofs that derive them from the course material.

[8] **1(a)** The reduced echelon form of the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 4 & 5 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1/2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 1 \end{bmatrix} \sim$

Compute the matrix product $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ -2 & 8 \end{bmatrix}$

Let A be a square matrix. It is defined to be invertible if ...

there is a matrix B so that $AB = I$, $BA = I$

$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/5 \end{bmatrix}$

[6] **1(b)** Find the set of solutions $\mathbf{x} \in \mathbb{R}^3$ of the equation $\begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 10 \end{bmatrix}$.

Then state the general solution principle for inhomogeneous linear equations of the form $T(\mathrm{x}) = \mathrm{b}$, specify T and b in this example, and explain how your result is an example of this principle.

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ -2 & 4 & 1 & -1 \\ 0 & 0 & 2 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 - 2x_2 = 3$$
$$x_3 = 5$$

solutions: $\left\{ \underline{x} = \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; x_2 \in \mathbb{R} \right\}$

<u>solution principle</u>: If $T: V \to W$ is linear, $b \in W$, $p \in V$ with $T(p) = b$, then

$$\left\{ \begin{matrix} \text{solutions} \\ \text{of } T(x) = b \end{matrix} \right\} = p + \left\{ \begin{matrix} \text{solutions} \\ \text{of } T(x) = 0 \end{matrix} \right\}$$

Here $T: \mathbb{R}^3 \to \mathbb{R}^3$ is given by multiplication with the matrix above, $\underline{b} = \begin{bmatrix} 3 \\ -1 \\ 10 \end{bmatrix}$, $p = \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ is a particular solution, and $\left\{ x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; x_2 \in \mathbb{R} \right\}$ is the set of solutions of $T(x) = 0$.

For work on this page to be graded, label it with problem number and write "XTRA" at that problem.

[6] **1(c)** Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 2x+y-z \\ 3y-z \\ 2z \end{bmatrix}$. $= A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Find a diagonal matrix D and a basis $\mathcal{B}$ of $\mathbb{R}^3$ so that the matrix for T relative to $\mathcal{B}$ is $[T]_{\mathcal{B}} = D$.

$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}$ has eigenvalues 3, 2 (double)

$\lambda = 3$ eigenspace $= \text{Nul} \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \text{Nul} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ spanned by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

or $\begin{bmatrix} c \\ c \\ 0 \end{bmatrix}$ with any $c \neq 0$

$\lambda = 2$ eigenspace $= \text{Nul} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{Nul} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(many other choices here)

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ yields $[T]_{\mathcal{B}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

alternative

$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ yields $[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

[8] **2(a)** A linear transformation $T : V \to W$ is defined to be one-to-one if ...

$T(x) = T(y)$ only for $x = y$

The span of two vectors v_1, v_2 in a general vector space V is defined to be ...

the set of linear combinations $c_1 v_1 + c_2 v_2$ with $c_1, c_2 \in \mathbb{R}$

Given the basis $\mathcal{B} = \{(1+t)^2, (1-t)^2, 1\}$ of $\mathbb{P}_2$, the $\mathcal{B}$-coordinates of $p(t) = 2t^2$ are

$$[p]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\begin{aligned} &\quad\ \, \| \\ &(1+t)^2 = 1 + 2t + t^2 \\ +&(1-t)^2 \quad +1 - 2t + t^2 \\ -&2 \cdot 1 \qquad -2 \end{aligned}$$

Find the matrix representation of the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2,\ p(t) \mapsto \frac{\mathrm{d}}{\mathrm{d}t} p(t)$ relative to the standard basis $\mathcal{B} = \{1, t, t^2\}$.

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} 1 &\mapsto 0 \rightsquigarrow \underline{0} \\ t &\mapsto 1 \rightsquigarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ t^2 &\mapsto 2t \rightsquigarrow \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \end{aligned}$$

[6] **2(b)** Recall that $\mathbb{R}^{2\times 2} = M_{2\times 2}$ is the vector space of 2×2 matrices with real entries. We define a map $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2},\ A \mapsto BA$ by multiplication with a fixed matrix $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

Show that T is linear and find a basis for its kernel. (Hint: This basis should consist of 2×2 matrices.)

<u>linearity</u>

- $T(A+W) = B(A+W) = BA + BW = T(A) + T(W)$
- $T(cA) = BcA = cBA$ for $c \in \mathbb{R}$

by properties of matrix multiplication

<u>kernel</u>: $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ 2a+2c & 2b+2d \end{bmatrix} = 0$

$\iff a+c=0,\ b+d=0$

$\text{kernel}(T) = \left\{\begin{bmatrix} a & b \\ -a & -b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}$

basis: $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$

<u>Alternative</u>: $A = [\underline{a}_1\ \underline{a}_2] \in \text{kernel}(T)$

$\iff BA = 0 \iff B\underline{a}_1 = \underline{0}$ and $B\underline{a}_2 = \underline{0}$

So from $\text{Nul}(B) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ we get

$\text{kernel}(T) = \left\{ A = \left[c\begin{bmatrix} -1 \\ 1 \end{bmatrix}\ d\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right] \,\middle|\, c, d \in \mathbb{R}\right\}$

$= \text{span}\left\{\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\right\}$

For work on this page to be graded, label it with problem number and write “XTRA” at that problem.

[6] 2(c) Let $T : \mathbb{R}^n \to V$ be a linear map to a general vector space V, and assume that $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$ is a basis of V. Use only definitions and algebra (no theorems) to prove that solutions of $T(\mathbf{x}) = v$ exist and are unique for every $v \in V$.

$$T(\underline{x}) = x_1 T(\underline{e}_1) + \ldots + x_n T(\underline{e}_n) = v$$

Finding $\underline{x}$, that is $x_1 .. x_n \in \mathbb{R}$, for a given v means finding the weights for writing v as linear combination of $T(\underline{e}_1), .., T(\underline{e}_n)$. So existence of solutions for all $v \in V$ is equivalent to $T(\underline{e}_1) \ldots T(\underline{e}_n)$ spanning V.

(Uniqueness of solutions implies in particular $T(\underline{x}) = 0$ only for $\underline{x} = \underline{0}$ (since $T(\underline{0}) = 0$ by linearity). Now $x_1 T(\underline{e}_1) + \ldots + x_n T(\underline{e}_n) = 0$ only for $\underline{x} = \underline{0}$ is exactly the definition of linear independence of $T(\underline{e}_1) \ldots T(\underline{e}_n)$)

Finally, linear independence of $T(\underline{e}_1) \ldots T(\underline{e}_n)$ implies that the solution of $T(\underline{x}) = 0$ is unique (namely $\underline{x} = \underline{0}$), and this implies uniqueness of solutions for all $v \in V$

$T(\underline{x}) = v$ and $T(\underline{y}) = v$ for any $v \in V$

$\Rightarrow T(\underline{x} - \underline{y}) = T(\underline{x}) - T(\underline{y}) = 0$ by linearity

$\Rightarrow \underline{x} - \underline{y} = \underline{0}$ by uniqueness for $v = 0$

$\Rightarrow \underline{x} = \underline{y}$

$r^2+1=0 \rightsquigarrow r=\pm i$

[8] **3(a)** The general solution of $y'' + y = 0$ is

$y(t) =$ $\boxed{c_1 \cos t + c_2 \sin t \quad \text{for } c_1, c_2 \in \mathbb{R}}$

$r^2 - 6r + 9 = 0$

The general solution of $y'' - 6y' + 9y = 0$ is $r = 3 \pm \sqrt{3^2 - 9} = 3$ (double)

$y(t) =$ $\boxed{c_1 e^{3t} + c_2 t e^{3t} \quad \text{with } c_1, c_2 \in \mathbb{R}}$

A (particular) solution of $y'' - 6y' + 9y = 3t + 7$ is

$y(t) =$ $\boxed{\frac{1}{3}t + 1}$

Ansatz: $y(t) = at + b$
$y' = a$
$y'' = 0$

$-6(a) + 9(at+b) = 3t + 7$

$\Leftrightarrow \begin{cases} 9a = 3 \\ -6a + 9b = 7 \end{cases} \Leftrightarrow \begin{cases} a = 1/3 \\ 9b = 7 + 6 \cdot \frac{1}{3} = 9 \end{cases}$

$\Downarrow$

$a = 1/3,\ b = 1$

Given two smooth functions $y_1(t), y_2(t)$ of $-\infty < t < \infty$, write "none", "$\Rightarrow$", "$\Leftarrow$", or "$\Leftrightarrow$" into the box for the implications between the following statements:

y_1, y_2 are linearly independent in $\mathcal{C}^\infty$ $\boxed{\Leftarrow}$ $\det \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \neq 0$ for some $t_0 \in \mathbb{R}$.

[6] **3(b)** Find the solution of $y'' + y = \sin 3t, \quad y(0) = 0, \quad y'(0) = 0.$
Hint: Use the first part of 3a).

Ansatz: $y(t) = A\cos 3t + B\sin 3t$

$y' = -3A\sin 3t + 3B\cos 3t$

$y'' = -9A\cos 3t - 9B\sin 3t$

Plug in: $y''+y = (\underbrace{-9A+A}_{-8A})\cos 3t + (\underbrace{-9B+B}_{-8B})\sin 3t = \sin 3t$

Solve: $A=0$, $B=\frac{-1}{8}$

general solution: $y(t) = -\frac{1}{8}\sin 3t + c_1 \sin t + c_2 \cos t$

initial conditions: $0 = y(0) = 0 + 0c_1 + c_2 \Leftrightarrow c_2 = 0$

$0 = y'(0) = -\frac{3}{8}\cos 3t\Big|_{t=0} + c_1\cos 0 - c_2 \sin 0$

$\Leftrightarrow c_1 - \frac{3}{8} = 0$

$\Rightarrow \underline{\underline{y(t) = \frac{-1}{8}\sin 3t + \frac{3}{8}\sin t}}$

For work on this page to be graded, label it with problem number and write “XTRA” at that problem.

[6] **3(c)** Find the general solution $y(t)$ of $T[y] = t^2$ and explain why there cannot be any other solutions, using only definitions, algebra, and the following information:

a) $T : \mathcal{C}^\infty \to \mathcal{C}^\infty$ is a linear transformation.

b) $T\left[t^2 + \frac{1}{2}\right] = -4t^2$.

c) The kernel of T is spanned by $y_1(t) = e^{2t}$, $y_2(t) = e^{-2t}$.

$$\left.\begin{matrix} a) \\ b) \end{matrix}\right\} \Rightarrow T\left[-\tfrac{1}{4}t^2 - \tfrac{1}{8}\right] = t^2$$

$$T[y] = t^2 \iff T[y] = T\left[-\tfrac{1}{4}t^2 - \tfrac{1}{8}\right]$$

$$\iff T\left[y + \tfrac{1}{4}\left(t^2 + \tfrac{1}{2}\right)\right] = 0$$

$$\iff y(t) + \tfrac{1}{4}\left(t^2 + \tfrac{1}{2}\right) = c_1 e^{2t} + c_2 e^{-2t} \quad \text{for some } c_1, c_2 \in \mathbb{R}$$

$$\iff y(t) = \tfrac{-1}{4}\left(t^2 + \tfrac{1}{2}\right) + c_1 e^{2t} + c_2 e^{-2t} \quad \text{for } c_1, c_2 \in \mathbb{R}$$

This is the general solution (i.e. they solve, and there are no other solutions) because each step above is an equivalence.

[8] **4(a)** The general solution of $\mathbf{x}' = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = \boxed{e^{3t}\begin{bmatrix} 5 \\ 1 \end{bmatrix} + e^{-2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

- eigenvalues $3, -2$
- -2-eigenspace spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- 3-eigenspace $= \mathrm{Nul}\begin{bmatrix} 0 & 0 \\ 1 & -5 \end{bmatrix}$ spanned by $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

If a real 2×2 matrix A satisfies $A\begin{bmatrix} 1 \\ i \end{bmatrix} = (2-3i)\begin{bmatrix} 1 \\ i \end{bmatrix}$ then the general solution of $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = \boxed{c_1 e^{2t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -\sin 3t \\ \cos 3t \end{bmatrix} \quad \text{for } c_1, c_2 \in \mathbb{R}}$$

$$e^{(2-3i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} = e^{2t}(\cos 3t - i \sin 3t)\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= e^{2t}\Bigg(\underbrace{\cos 3t\begin{bmatrix} 1 \\ 0 \end{bmatrix} - i^2 \sin 3t\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} \rightsquigarrow \mathrm{Re}} + i\big(\underbrace{\cos 3t\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin 3t\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\begin{bmatrix} -\sin 3t \\ \cos 3t \end{bmatrix} \rightsquigarrow \mathrm{Im}}\big)\Bigg)$$

The fundamental matrix $\mathbf{X}(t)$ of an ODE system $\mathbf{x}' = A\mathbf{x}$ is defined to be ...

> any matrix function that solves $\mathbf{X}' = A\mathbf{X}$ and $\det \mathbf{X}(t) \neq 0$ for some (and hence all) $t \in \mathbb{R}$

alternative: $\mathbf{X}(t) = [\underline{x}_1(t) \dots \underline{x}_n(t)]$ where $\underline{x}_1 \dots \underline{x}_n$ is a basis of the set of solutions

[6] **4(b)** Find the solution $\mathbf{x}(t)$ of $\mathbf{x}' = \begin{bmatrix} 1 & 5 \\ 1 & -3 \end{bmatrix}\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$.

- eigenvalues: $\det \begin{bmatrix} 1-\lambda & 5 \\ 1 & -3-\lambda \end{bmatrix} = (\lambda-1)(\lambda+3) - 5$

$$= \lambda^2 + 2\lambda - 3 - 5 = 0$$

$$\Leftrightarrow \lambda = -1 \pm \sqrt{1^2+8} = -1 \pm 3 = -4, 2$$

- eigenvectors:

$\lambda = -4$: $\mathrm{Nul}\begin{bmatrix} 1+4 & 5 \\ 1 & -3+4 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 5 & 5 \\ 1 & 1 \end{bmatrix}$ spanned by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 2$: $\mathrm{Nul}\begin{bmatrix} 1-2 & 5 \\ 1 & -3-2 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} -1 & 5 \\ 1 & -5 \end{bmatrix}$ spanned by $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$\Rightarrow$ general solution: $\underline{x}(t) = c_1 e^{-4t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

- initial value: $\underline{x}(0) = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

$$\Leftrightarrow \begin{matrix} -c_1 + 5c_2 = 0 \\ c_1 + c_2 = 6 \end{matrix} \Leftrightarrow \begin{matrix} c_1 = 5c_2 \\ 5c_2 + c_2 = 6 \end{matrix} \Leftrightarrow \begin{matrix} c_1 = 5 \\ c_2 = 1 \end{matrix}$$

$$\Rightarrow \underline{x}(t) = \underline{\underline{5e^{-4t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + e^{2t}\begin{bmatrix} 5 \\ 1 \end{bmatrix}}}$$

For work on this page to be graded, label it with problem number and write "XTRA" at that problem.

[6] **4(c)** Show that $\mathbf{x}_1(t) = \begin{bmatrix} 2t^2 \\ 6t \end{bmatrix}$, $\mathbf{x}_2(t) = \begin{bmatrix} t^3 \\ 3t^2 \end{bmatrix}$ are linearly independent as $\mathbb{R}^2$-valued functions. Compute their Wronskian, and use it to explain whether $\mathbf{x}_1, \mathbf{x}_2$ can solve the same system $\mathbf{x}' = A\mathbf{x}$.

$$c_1 \underline{x}_1(t) + c_2 \underline{x}_2(t) = 0 \quad \text{for all } t$$

$$\Leftrightarrow \quad \begin{aligned} 2c_1 t^2 + c_2 t^3 &= 0 \\ 6c_1 t + 3c_2 t^2 &= 0 \end{aligned} \quad \text{for all } t$$

$$\Rightarrow \quad \begin{aligned} 2c_1 + c_2 &= 0 \quad (\text{from } t=1) \\ -6c_1 + 3c_2 &= 0 \quad (\text{from } t=-1) \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} c_2 &= -2c_1 \\ c_2 &= 2c_1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2c_2 &= 0 \\ c_2 &= 2c_1 \end{aligned} \quad \Rightarrow \quad c_1 = c_2 = 0$$

Shows linear independence

Wronskian: $\det \begin{bmatrix} 2t^2 & t^3 \\ 6t & 3t^2 \end{bmatrix} = 2t^2 \cdot 3t^2 - t^3 \cdot 6t = 0$ for all t

$\underline{x}_1, \underline{x}_2$ cannot solve the same ODE system $\underline{x}' = A\underline{x}$ because if they did, then $W(t) = 0$ for some $t \in \mathbb{R}$ would imply (by "Wronskian Theorem") that $\underline{x}_1, \underline{x}_2$ are linearly dependent.

[6] **5(b)** Find the solution $u(t,x)$ for $0 < x < \pi,\ t > 0$ to the initial-boundary value problem

$$\frac{\partial u}{\partial t} = 4\,\frac{\partial^2 u}{\partial x^2}, \qquad \underbrace{\frac{\partial u}{\partial x}(t,0) = 0, \qquad u(t,\pi) = 0,} \qquad u(0,x) = \cos(\tfrac{1}{2}x) - \cos(\tfrac{7}{2}x).$$

- Fourier Ansatz: $u(t,x) = \sum_{n=0}^{\infty} c_n(t) \cos(n+\frac{1}{2})x$

CHECK: $\partial_x u|_{x=0} = \sum c_n(t)\,(-n-\frac{1}{2}) \sin 0 = 0$

$u|_{x=\pi} = \sum c_n(t) \cos(n+\frac{1}{2})\pi = 0$

- plug into PDE: $\partial_t u = \sum c_n'(t) \cos(n+\frac{1}{2})x$

$$\begin{array}{c} \parallel \\ 4\partial_x^2 u = \sum 4\, c_n(t)\,(-1)\,(n+\tfrac{1}{2})^2 \cos(n+\tfrac{1}{2})x \end{array}$$

$\Longleftrightarrow \quad c_n' = -4(n+\frac{1}{2})^2 c_n$

- plug into initial conditions: $\sum_{n=0}^{\infty} c_n(0) \cos(n+\frac{1}{2})x = \cos\frac{1}{2}x - \cos\frac{7}{2}x$

$\Longleftrightarrow \ c_0(0) = 1,\ c_3(0) = -1,$ other $c_n = 0$

$(3+\frac{1}{2} = \frac{7}{2})$

- solve: $c_n(t) = 0$ except for

<u>$n=0$</u>: $c_0' = -4\,(\frac{1}{2})^2\, c_0 = -c_0 \quad \Rightarrow \quad c_0(t) = c_0(0)\, e^{-t} = e^{-t}$

<u>$n=3$</u>: $c_3' = -4\,(3+\frac{1}{2})^2\, c_3 = -49\, c_3 \quad \Rightarrow \quad c_3(t) = c_3(0)\, e^{-49t} = -e^{-49t}$

- plug back:

$$u(t,x) = \underline{\underline{e^{-t} \cos\tfrac{1}{2}x - e^{-49t} \cos\tfrac{7}{2}x}}$$

[8] **5(a)** Fill in the ... below.

The Fourier Sine series of a continuous function $f : [0, 100] \to \mathbb{R}$ is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(n \frac{\pi}{100} x\right)$$

with coefficients given by the inner product formula

$$b_n = \frac{\langle f, \sin(n \frac{\pi}{100} x) \rangle}{\| \sin(n \frac{\pi}{100} x) \|^2}$$

These coefficients are also given by the integral $\int_0^{100} \sin(n \frac{\pi}{100} x)^2 \, dx = 100 \cdot \frac{1}{2} = 50$

$$b_n = \frac{1}{50} \int_0^{100} f(x) \sin\left(n \frac{\pi}{100} x\right) dx$$

The coefficients (as defined above) of the Fourier Sine series of $f(x) = \sin \pi x$ on $[0, 100]$ are

$= 1 \cdot \sin\left(100 \cdot \frac{\pi}{100} x\right)$

$b_n = 0$ except for $b_{100} = 1$

The Fourier Sine series of $f(x) = 1$ on $[0, 100]$ is

$$1 \sim \sum_{n \text{ odd}} \frac{4}{\pi n} \sin\left(n \frac{\pi}{100} x\right) = \sum_{k=0}^{\infty} \frac{4}{\pi (2k+1)} \sin\left((2k+1) \frac{\pi}{100} x\right)$$

$$\int_0^{100} 1 \cdot \sin\left(n \frac{\pi}{100} x\right) dx = \left[\frac{-100}{\pi n} \cos\left(n \frac{\pi}{100} x\right) \right]_0^{100} = \frac{100}{\pi n} \left(\underbrace{\cos 0 - \cos n\pi}_{\substack{= 1 - 1 \text{ for } n \text{ even} \\ = 1 + 1 \text{ for } n \text{ odd}}} \right)$$

n even : $b_n = 0$

$\Rightarrow$ n odd : $b_n = \frac{1}{50} \cdot \frac{100}{\pi n} \cdot 2 = \frac{4}{\pi n}$

For work on this page to be graded, label it with problem number and write “XTRA” at that problem.

[6] 5(c) Consider the PDE problem for a function $u(t,x,y)$ of $0<x<\pi, 0<y<\pi, t>0$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x}(t,0,y)=0, \quad \frac{\partial u}{\partial x}(t,\pi,y)=0, \quad u(t,x,0)=0, \quad u(t,x,\pi)=0.$$

Find the general (formal) solution of the form $u(t,x,y)=\sum_{m=\dots}\sum_{n=\dots} C_{m,n}(t)X_m(x)Y_n(y)$ by specifying $C_{m,n}, X_m, Y_n$ up to some constants. Show your work or otherwise check that your solution solves the problem.

$$\sum C_{m,n}(t)X_m'(0)Y_n(y)=0 \qquad\qquad \sum C_{m,n}(t)X_m'(\pi)Y_n(y)=0$$

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$$X_m'(0)=0 \qquad\qquad X_m'(\pi)=0$$

$$\Leftarrow \qquad\qquad \Rightarrow$$

$$\underline{X_m(x)=\cos mx \qquad m=0,1,2,\dots}$$

$$0=u(t,x,0)=\sum C_{m,n}(t)X_m(x)Y_n(0) \Leftarrow Y_n(0)=0$$

$$0=u(t,x,\pi)=\sum C_{m,n}(t)X_m(x)Y_n(\pi) \Leftarrow Y_n(\pi)=0$$

$$\underline{Y_n(y)=\sin ny}$$
$$\underline{n=1,2,\dots}$$

$$\partial_t^2 u = \sum C''_{m,n}\cos mx \sin nx$$

$$\|$$

$$\partial_x^2 u+\partial_y^2 u = \sum C_{m,n}(-m^2)\cos mx \sin nx + \sum C_{m,n}\cos mx(-n^2)\sin nx$$

is satisfied if for all m,n

$$C''_{m,n}(t) = -(m^2+n^2)C_{m,n}(t)$$

$$\Leftrightarrow C_{m,n}(t) = a_{m,n}\cos\sqrt{m^2+n^2}\,t + b_{m,n}\sin\sqrt{m^2+n^2}\,t$$

with $a_{m,n}, b_{m,n}\in\mathbb{R}$

$\Rightarrow$ formal solution

$$u(t,x,y)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(a_{m,n}\cos\sqrt{m^2+n^2}\,t+b_{m,n}\sin\sqrt{m^2+n^2}\,t\right)\cos mx\cdot\sin ny$$

For work on this page to be graded, label it with problem number and write “XTRA” at that problem.