Midterm 2

Last Name	First Name	SID

- You have 10 minutes to read the exam and 100 minutes to complete this exam.
- The maximum you can score is 100.
- The exam is not open book, but you are allowed to consult the cheat sheet that we provide. No calculators or phones. No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- **A correct answer without justification will receive little, if any, credit.**
- Take into account the points that may be earned for each problem when splitting your time between the problems.

Problem	points earned	out of
Problem 1		50
Problem 2		20
Problem 3		20
Problem 4		30
Total		100 (+20)

Problem 1: Answer these questions briefly but clearly. [50]

1. **Chernoff Bound [8]**

   Suppose X is Gaussian with mean zero and unit variance. We want to obtain an upper-bound on $\mathbb{P}(X > t)$ for $t > 0$. Use Chernoff's bound which involves $M_X(s)$, the MGF of X, and optimize this bound over s to make it as tight as possible. Recall that the MGF is defined as $M_X(s) = \mathbb{E}[e^{sX}]$.

   For $s > 0$, we have

$$\begin{aligned}\mathbb{P}(X > t) &= \mathbb{P}(e^{sX} > e^{st})\\ &\leq \frac{\mathbb{E}(e^{sX})}{e^{st}}\\ &= M_X(s)e^{-st}\\ &= e^{\frac{s^2}{2}-st}\end{aligned}$$

   In order to minimize this bound, we need to minimize $s^2/2 - st$, which is achieved at $s = t$. Substituting for $s = t$, we arrive at the bound

$$\mathbb{P}(X > t) \leq e^{-t^2/2}.$$

2. **Coins and Markov Chains [8]**

   You lay out n coins on a table, some of which face heads up, and some of which face tails up, at time zero. Then, at each time-step, with some small probability r, you do nothing. Otherwise, you pick a coin uniformly at random from the n coins, and flip it over (so if it was facing heads earlier, it will now face tails). We wish to model X_t, the number of coins that face heads up at time t. The process is allowed to continue for N iterations, for some very large N, and then we stop the process. Estimate the probability that there are k coins facing heads.

   The states are $\{0, 1, 2, ..., n\}$. There is a self loop on every state with probability r. Otherwise, $\mathbb{P}(X_{t+1} = k+1 | X_t = k) = (1-r)((n-k)/n)$ and $\mathbb{P}(X_{t+1} = k-1 | X_t = k) = (1-r)(k/n)$ This is an irreducible finite Markov chain with self loops and is hence also aperiodic. We want to find the stationary distribution $\pi(k)$ of the chain. We set up detailed balance equations as

$$\frac{(1-r)k}{n}\pi(k) = \frac{(1-r)(n-k+1)}{n}\pi(k-1) \implies \pi(k) = \frac{n-k+1}{k}\pi(k-1)$$

   Unroll this to get $\pi(k) = \binom{n}{k}\pi(0)$. To find $\pi(0)$, we can enforce that $\sum \pi(i) = 1$. Then we have $\pi(0)\sum_{i=0}^{n}\binom{n}{i} = 1 \implies \pi(0) = 2^{-n}$. Finally, we get $\pi(k) = \binom{n}{k}2^{-n}$. So finally, we can say that for very large N, the probability that k coins show heads is $\binom{n}{k}2^{-n}$.

3. **Got the Message? [8]**

We want to send one of 2^n equally likely messages reliably over a Binary Erasure Channel (BEC) with probability of erasure $p = 0.5$. For the sake of simplicity, assume that exactly a fraction p of the input bits are erased through the channel, though you are not told which ones.

(a) What is a tight upper bound on n if I am allowed 10,000 independent uses of the channel? Give an argument for why this bound cannot be exceeded.

Even assuming that at the decoder, we know which bits were erased, since half of the bits are erased, we can not transmit more than $p \times 10,000 = 5000$ bits. Hence, n can not be bigger than 5000.

(b) If I use Shannon's random coding scheme, what n is achievable, assuming that we want to have a probability of success greater than $1 - 2^{-100}$?

Recalling the error analysis for Shannon's random coding scheme, probability of error is upper bounded by $2^n \times 2^{-5000}$ which must be smaller than 2^{-100}. This means that $n - 5000 < -100$ or equivalently $n < 4900$.

4. **Estimate of Rate for Poisson Process [8]**

Consider a Poisson process of unknown rate λ. You have access to observations $t_1, t_2, ..t_n$, where t_i represents the time i^{th} arrival happens (starting from some arbitrary $t = 0$). Compute the MLE of λ.

We know the interarrival times $\Delta t_i = t_{i+1} - t_i$ are distributed i.i.d exponentially. So we write the required PDF

$$
\begin{gathered}
f\left(T_1 = t_1, \ldots T_n = t_n\right) \\
= f\left(T_1 = t_1, \Delta T_1 = \Delta t_1, \ldots \Delta T_n = \Delta t_n\right) \\
= \left(\lambda e^{-\lambda t_1}\right) \cdot \left(\lambda e^{-\lambda(t_2 - t_1)}\right) \cdot \left(\lambda e^{-\lambda(t_3 - t_2)}\right) \ldots \left(\lambda e^{-\lambda(t_n - t_{n-1})}\right) \\
= \lambda^n e^{-\lambda t_n}
\end{gathered}
$$

Taking the derivative w.r.t λ and setting it to 0 we get

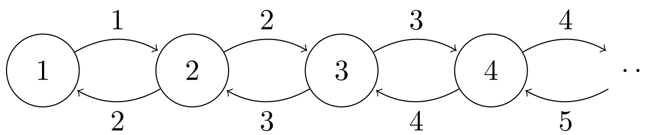
$$n\lambda^{n-1} e^{-\lambda t_n} - \lambda^n t_n e^{-\lambda t_n} = 0$$

Solving for λ we get

$$\lambda = \frac{n}{t_n}$$

5. **Infinite CTMC [18]**

Consider a CTMC with states $\{1, 2, \ldots\}$ characterized by the following rate transition diagram below. Is it positive or null recurrent? Provide justification. (Hint: $\sum_{i=1}^{\infty} \frac{1}{i}$ is a divergent series.)



The chain is null recurrent. An irreducible, possibly infinite Markov chain is positive recurrent if and only if it has a stationary distribution. Let's attempt to calculate the stationary distribution. Using the flow balance equations, we get that

$$\pi(i-1) \times (i-1) = \pi(i) \times i$$

This implies that

$$\begin{aligned}
\pi(i) &= \frac{i-1}{i}\pi(i-1) \\
&= \frac{i-1}{i} \times \frac{i-2}{i-1} \times \cdots \times \frac{2}{3} \times \frac{1}{2}\pi(1) \\
&= \frac{1}{i}\pi(1)
\end{aligned}$$

We can calculate $\pi(1)$ because we know that

$$\begin{aligned}
1 &= \sum_{i=1}^{\infty} \pi(i) \\
&= \pi(1) \sum_{i=1}^{\infty} \frac{1}{i}
\end{aligned}$$

However, since $\sum_{i=1}^{\infty} \frac{1}{i}$ is a divergent series $\pi(1)$ does not exist.

Problem 2: Erdös-Rényi Random Graphs & Poisson Processes (The most ambitious crossover in history?) [20]

Consider a set of N vertices. For each vertex, assume that there exists a Poisson arrival process of rate λ independent of the arrival processes of the other vertices (λ is the same for all vertices). Once there is an arrival at a vertex, that arrival is routed to one of the other $N-1$ vertices, chosen uniformly at random, and *discarded at that vertex*. Assume that this routing is done instantaneously.

Now, let T be a fixed length of time. We wait for T seconds, and then draw an edge (v, w) between vertices v and w if and only if the sum of the number of arrivals that are routed from v to w and from w to v is at least some positive integer k during the time interval $[0, T]$.

(a) Find the probability that a particular edge exists in the graph. You may have summations in your answer.

Let v, w be arbitrary vertices in the graph. By Poisson splitting, the times at which an arrival is routed from v to w follows a Poisson process of rate $\lambda/(N-1)$. The same holds for the arrivals that are routed from w to v, and these two processes are independent. Therefore, by Poisson merging, the times at which arrivals are routed $u \to v$ or $v \to u$ is a Poisson process of rate $2\lambda/(N-1)$. Since the number of arrivals of a Poisson process of rate μ that occurs in the interval $[0, T]$ follows a Poisson distribution with parameter μT, the probability that a particular edge exists in the graph is given by

$$P((v, w) \text{ exists}) = 1 - \sum_{i=0}^{k-1} \frac{1}{i!} \left(\frac{2\lambda T}{N-1} \right)^i \exp\left[-\frac{2\lambda T}{N-1} \right].$$

(b) Prove or disprove: The above construction is an Erdös-Rényi random graph, in the sense defined in class. (Remember: For Erdös-Rényi random graphs, each edge exists independently of all others, and the probability of an edge existing is the same across all edges).

The construction does result in an Erös-Rényi random graph. We have shown in (a) that the probability that an edge exists is the same regardless of which edge is being considered. Therefore, we simply have to show that the event that some edge exists is independent of all the presence of (or lack of) other edges. Since Poisson splitting results in independent Poisson processes, and the Poisson processes assigned to each vertex were assumed to be independent of each other, the two Poisson processes contributing to a particular edge is independent of the Poisson processes for other edges. Therefore, the edges are independent of each other, and the construction indeed results in an Erdös-Rényi random graph.

Problem 3: Trick-or-treat Questions for Poisson Processes [20]

1. **Sum of Waiting Times [12]**

   Suppose that Halloween customers arrive at a spooky costume store in accordance with a Poisson process with rate λ. If the shopkeeper arrives at time t, compute the expected sum of waiting times of the customers arriving in $(0, t)$, conditioned on the fact that there are 10 arrivals in $(0, t)$. That is, compute $\mathbb{E}\left(\sum_{i=1}^{N(t)}(t - S_i)\Big| N(t) = 10\right)$, where S_i is the arrival time for the i th customer and $N(t)$ is the total number of customers in the interval $(0, t)$.

   We know that conditioned on $N(t) = 10$, $S_1, \ldots, S_{10}$ are the order statistics of 10 independent random variables $U_1, \ldots, U_{10}$ where U_i is uniformly distributed in $(0, t)$. More specifically, we have $S_i = U_{(i)}$ for $1 \leq i \leq 10$, where $U_{(i)}$ denoted the i–th smallest among $U_1, \ldots, U_{10}$. Therefore, we have

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{N(t)}(t - S_i)|N(t) = 10\right) &= \mathbb{E}\left(\sum_{i=1}^{10}(t - U_{(i)}\right) \\
&= \mathbb{E}\left(\sum_{i=1}^{10} t - U_i\right) \\
&= \sum_{i=1}^{10} \mathbb{E}\left(t - U_i\right) \\
&= \sum_{i=1}^{10} t/2 = 5t.
\end{aligned}$$

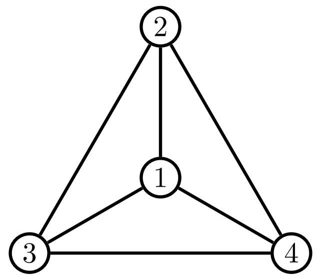
   Here, at the second line, we have used the fact that the order of U_i's does not matter in computing the sum.

2. **Lazy Shopkeeper [8]**

   A customer arrives at a store at 1 pm. The shopkeeper is lazy and instead of being in the store all the time, visits the store in a Poisson process with rate $\lambda = 1$ per hour, independent of the arrival time of customer. When the shopkeeper arrives, he gives a discount to the customer linearly proportional to the time since his previous visit to the shop, and the amount of this discount is ten dollars per hour. What is the expected discount the shopkeeper needs to give to the customer? Assume that the shopkeeper has been visiting the store since a very long time ago.

   Recalling the random incidence phenomenon, the expectation of the interval between arrivals of the shopkeeper from the point of view of the customer is twice the expectation of inter–arrival time, which becomes $2 \times 1 = 2$ hours. Therefore, the customer receives 20 dollars discount in expectation.

Problem 4: Joint Random Walk [30] Alice walks on the following graph randomly, that is, if X_k is her position at time k, which is one of the values $\{1, 2, 3, 4\}$, then her position at time $k+1$ is chosen uniformly at random among the neighbors of X_k (not including X_k itself, i.e, you can not stay at the same node on consecutive time steps).



(a) Prove or disprove: The Markov chain $(X_k : k \geq 0)$ has a unique stationary distribution. [4]

Note that X_k is irreducible and finite. Hence, it has a unique stationary distribution.

(b) If $X_0 = 1$, what is the long time frequency that Alice visits state 2? [6]

Moreover, since all the vertices have degree 3, the uniform distribution is this stationary distribution. To check this, one can verify the detailed balanced equations

$$\pi(i)p_{i,j} = \frac{1}{4} \times \frac{1}{3} = \pi(j)p_{j,i}.$$

Consequently, Alice spends $1/4$ of her time in state 2.

(c) Now, assume that Bob joins Alice and they do the above random walk on the graph independently, where X_k and Y_k denote Alice's and Bob's position at time k, respectively. Assume that both of them are initially at state 1, i.e. $X_0 = Y_0 = 1$, find $\lim_{k\to\infty} \mathbb{P}(X_k = Y_k)$. [8]

We have

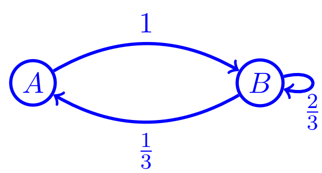
$$\mathbb{P}(X_k = Y_k) = \sum_{i=1}^{4} \mathbb{P}(X_k = Y_k = i) = \sum_{i=1}^{4} \mathbb{P}(X_k = i)\mathbb{P}(Y_k = i),$$

where the last equality follows from the independence of X_k and Y_k. But form the previous part, we know that as $k \to \infty$, $\mathbb{P}(X_k = i)$ and $\mathbb{P}(Y_k = i)$ go to $\pi(i) = 1/4$ for all $1 \le i \le 4$. Hence, we have

$$\lim_{k\to\infty} \mathbb{P}(X_k = Y_k) = 4 \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{4}.$$

(d) Similar to the previous part, assume that Alice and Bob start at position 1, and walk independently. What is the expectation of the first time that one of them is in the center node 1 and the other one is in one of the other nodes $\{2, 3, 4\}$? [12]
[Hint: use symmetry to simplify the problem. Specifically, you can reduce the random walk for each of the two players to 2 states which captures all the information needed for the purpose of finding the expectation.]

Define a 2 state Markov chain $\tilde{X}_k$ on two states $\{A, B\}$ as follows: let $\tilde{X}_k = A$ if $X_k = 1$, and $\tilde{X}_k = B$ if $X_k \in \{2, 3, 4\}$. Similarly, define $\tilde{Y}_k$ based on Y_k. We want to find the expectation of the minimum k such that $(\tilde{X}_k, \tilde{Y}_k) \in \{(A, B), (B, A)\}$. The transitions of this new Markov chain is as follows:



Let x_{AA} be the above expectation given $(\tilde{X}_0, \tilde{Y}_0) = (A, A)$. Define x_{AB}, x_{BA} and x_{BB} in a similar fashion. Observer that by definition, $x_{AB} = x_{BA} = 0$. Writing first step equations, we get

$$x_{AA} = 1 + x_{BB}$$
$$x_{BB} = 1 + \frac{4}{9} x_{BB} + \frac{1}{9} x_{AA}.$$

Solving for this, we get $x_{AA} = \frac{7}{2}$.

Alternative solution: Alternatively, we can directly work with the above Markov chains. Let $a_{i,j}$ denote the mentioned expectation given that $X_0 = i$ and $Y_0 = j$. We may exploit symmetry in the problem to write down first step equations to find the desired value $a_{1,1}$. First note that $a_{i,j} = a_{j,i}$. Furthermore, we have

$$a_{1,2} = a_{1,3} = a_{1,4} = 0$$

$$
\begin{aligned}
a_{2,2} &= a_{3,3} = a_{4,4} =: y \\
a_{2,3} &= a_{2,4} = a_{3,4} =: z \\
a_{1,1} &=: x
\end{aligned}
$$

Writing first step equations, we have

$$
\begin{aligned}
x &= 1 + \frac{1}{3}y + \frac{2}{3}z \\
y &= 1 + \frac{1}{9}x + \frac{2}{9}y + \frac{2}{9}z \\
z &= 1 + \frac{1}{9}x + \frac{1}{3}z + \frac{1}{9}y
\end{aligned}
$$

Solving for this, we get $x = 7/2$ which is the same as the previous solution.