Mathematics 54 — Midterm 2, 11/6/19

50 minutes, 50 points

Question 1. (15 points) Bubble in the correct answers, worth 1 point each. No justification necessary. Incorrect answers carry a 1-point penalty, so **random choices are not helpful**. You may leave any question blank for 0 points. You will not get a negative score on any group of five questions.

T An orthonormal collection of vectors in $\mathbb{R}^n$ is linearly independent.
T All eigenvalues of a symmetric matrix are real.
T Every linear system has a Least Squares solution.
F A square matrix is invertible precisely when 0 is an eigenvalue of it.
T If the columns of the real 2×2 matrix $M \neq I_2$ are orthonormal, then the matrix transformation $\mathbf{x} \mapsto M\mathbf{x}$ is a rotation or a reflection in $\mathbb{R}^2$.

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T The $n \times n$ matrix representing the orthogonal projection onto a line in $\mathbb{R}^n$ has rank one.
T The product of two orthogonal matrices of the same size is also an orthogonal matrix.
T The Least-Squares problem is finding a vector $\mathbf{x}$ which makes $A\mathbf{x}$ as close as possible to a given vector $\mathbf{b}$.
F If $(t-2)$ is a factor of the characteristic polynomial of A, then (-2) is an eigenvalue of A.
T The determinant of a square matrix is the product of its eigenvalues, included with their multiplicities.

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T If $\|\mathbf{u} - 2\mathbf{v}\| = \|\mathbf{u} + 2\mathbf{v}\|$, then the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
T If a vector $\mathbf{v}$ is orthogonal to every column of the matrix A, then $\mathbf{v}^T$ is in the left nullspace of A.
F Similar matrices have the same eigenvectors.
T If $AS = S$, then every nonzero column of S is an eigenvector of A.
T If A^TA is the identity matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ preserves lengths: $\|\mathbf{x}\| = \|A\mathbf{x}\|$ for all $\mathbf{x}$.

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T If a square matrix has orthonormal columns, then it also has orthonormal rows.
F* If W is a subspace of $\mathbb{R}^n$ and $\mathbf{p}$ is in W and $\mathbf{q}$ in $W^\perp$, then $\|\mathbf{p} - \mathbf{q}\| = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2$.
F Every upper triangular matrix A is diagonalizable.
T A real 2×2 matrix which has one *non-real* eigenvalue must be diagonalizable over $\mathbb{C}$.
T If the columns of the $m \times n$ matrix A are linearly independent, then the matrix A^TA is invertible.

* In an unfortunate typo, the square is missing in Pythagoras. We must go with what is written and the statement is false.

Question 2. (11 points, 2+3+3+3)
Determine, for the matrix

$$A = \begin{bmatrix} 0.7 & 0.15 \\ 0.4 & 0.8 \end{bmatrix},$$

(a) the characteristic polynomial; (b) the eigenvalues; (c) an eigenbasis of $\mathbb{R}^2$; (d) a formula for A^n.

$\chi_A(t) = t^2 - 1.5t + 0.5$ so the eigenvalues are 1 and .5.
An eigenvector for 1 is $[1, 2]^T$ and for .5 $[-3, 4]^T$. They form an eigenbasis, and

$$A = \frac{1}{10} \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$$

so that

$$A^n = \frac{1}{10} \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2^n \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 4 + 6/2^n & 3(1 - 1/2^n) \\ 8(1 - 1/2^n) & 6 + 4/2^n \end{bmatrix}$$

Question 3. (12 points, 6+4+2)
For the vector $\mathbf{x} = [1, 2, 2, 3]^T$ and the subspace V of $\mathbb{R}^4$ defined by the equations

$$x_1 - x_2 + x_3 - x_4 = 0 \quad \text{and} \quad 3x_1 + x_2 - x_3 - 3x_4 = 0,$$

(a) Find the 4×4 matrix implementing the orthogonal projection of $\mathbb{R}^4$ onto V;
(b) Find the orthogonal projection of $\mathbf{x}$ onto V.
(c) Find the distance from $\mathbf{x}$ to V.

Project to $V^\perp$ fir which webalready have the basis $[1, -1, 1, -1]^T, [3, 1, -1, -3]^T$. Assembling them as columns of a matrix A, the formula for the projection matrix onto $V^\perp$ reads

$$A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 1 & -1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ 4 & 20 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 3 & 1 & -1 & -3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

So the projection onto V is given by subtracting this from I_4:

$$P = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}; \quad P\mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}; \quad \mathbf{x} - P\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

For a check, you can see that $\mathbf{x} \in V$ and $\mathbf{x} - P\mathbf{x}$ is orthogonal to V (because $x_1 = x_4$ for all vectors in V). The distance from $\mathbf{x}$ to V is the length of this last vector which is $\sqrt{2}$.

Question 4. (12 points)
Set up and solve a consistent system of linear equations for the coefficients c_0, c_1 and c_2, whose solution gives the best fit to the relation $y = c_0 + c_1x + c_2(x^2 - x)$, in the sense of least squares, with the following data points:

x	-1	0	1	2
y	1	0	1	2

Plugging in the data for x, y gives the (inconsistent) system of equations

$$c_0 - c_1 + 2c_2 = 1; c_0 = 0; c_0 + c_1 =; c_0 + 2c_1 + 2c_2 = 2.$$

Writing A for the coefficient matrix of the system, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 6 & 2 \\ 4 & 2 & 8 \end{bmatrix}$$

Writing $\mathbf{b}$ for the vector of y-values, the normal equations in the unknown vector $\mathbf{c} = [c_0, c_1, c_2]^T$, $A^TA\mathbf{c} = A^T\mathbf{b}$, become

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 6 & 2 \\ 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

giving $c_0 = 0.3, c_1 = 0.4, c_2 = 0.5$.