Mathematics 54
Final Exam, 16 December 2019
180 minutes, 90 points

Question 1. (35 points) Select the correct answers, for 2.5 points each. No justification needed. Incorrect answers carry *no penalty* (but also no credit).

1. When can we be certain that a system $A\mathbf{x} = \mathbf{b}$, with a 5×4 matrix A, is consistent?

(a) Always
(b) When $\mathbf{b}$ is in $\mathrm{Nul}(A)$
(c) When $\mathbf{b} \perp \mathrm{Nul}(A)$
(d) When $\mathbf{b} \perp \mathrm{LNul}(A)$
(e) When A has four pivots
(f) When $\mathrm{Nul}(A) = \{\mathbf{0}\}$

2. For which vector $\mathbf{b}$ below does the system $\begin{bmatrix} 2 & 4 \\ 4 & 6 \\ 3 & 4 \end{bmatrix} \mathbf{x} = \mathbf{b}$ have a solution?

(a) $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (f) $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

3. For general matrix A, which of the following must remain *unchanged* under row operations?
(i) The row space (ii) The column space (iii) The positions of the pivot columns

(a) (i) and (ii)
(b) (ii) and (iii)
(c) (i) and (iii)
(d) (i), (ii) and (iii)
(e) (ii) but not (i) or (iii)
(f) All of them can change

4. Which of the following collections of vectors in $\mathbb{R}^4$ are linearly dependent?

(a) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4$
(b) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_1$
(c) $\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_3 - \mathbf{e}_4$
(d) $\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_3 - \mathbf{e}_1$
(e) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$
(f) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_4$

5. Which of the matrices below have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 3 & 2 \\ 4 & 7 & 7 & 4 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 0 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 7 & 10 \end{bmatrix}; \quad D = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

(a) A, B and C but not D
(b) B and C but not A, D
(c) A and C but not B, D
(d) C and D but not A, B
(e) B, C and D but not A
(f) They all have rank 2

6. For a general $m \times n$ matrix A, the dimensions of $\mathrm{Col}(A)$ and of $\mathrm{Row}(A)$ agree *if and only if*

(a) A is symmetric
(b) A is square
(c) A is diagonalizable
(d) A is invertible
(e) They always agree!
(f) A is orthogonal

7. If A and B are square matrices of the same size, we can safely conclude that

(a) $AB = BA$
(b) $(A - B)(A + B) = A^2 - B^2$
(c) $AB^T = B^T A$
(d) $(AB)^T = B^T A^T$
(e) $(AB)^T = A^T B^T$
(f) None of the above.

8. If linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $T(\mathbf{e}_1 - \mathbf{e}_2) = \mathbf{e}_2 - \mathbf{e}_3$, $T(\mathbf{e}_2 - \mathbf{e}_3) = \mathbf{e}_3 - \mathbf{e}_1$ and $T(\mathbf{e}_3 - \mathbf{e}_1) = \mathbf{e}_1 - \mathbf{e}_2$, then we can be certain that

(a) T is invertible

(b) T is orthogonal

(c) $T(\mathbf{e}_1) = \mathbf{e}_2$

(d) $\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$ is in the range of T

(e) T has rank 2 or more

(f) T does not exist

9. The following is an eigenvalue of $A = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 1 & 5 \\ 0 & 0 & 7 \end{bmatrix}$:

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5 (f) (-1)

10. Let A be a 3×4 matrix. Which of the following statements about $A^T A$ *cannot* be true?

(a) It is square

(b) It is symmetric

(c) It is invertible

(d) It has rank 3

(e) It is diagonalizable over $\mathbf{R}$

(f) Its eigenvalues are ≥ 0

11. In which situation below can we be sure that the real $n \times n$ matrix A has *positive* determinant?

(a) A has positive entries

(b) There exists a matrix B with $AB = I_n$

(c) A has positive pivots

(d) A is diagonalizable

(e) All eigenvalues of A are positive real

(f) A is orthogonal

12. The least-squares solution to $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ is

(a) $\mathbf{x} = 0$ (b) $\mathbf{x} = 1$ (c) $\mathbf{x} = 2$ (d) $\mathbf{x} = 3$ (e) $\mathbf{x} = 4$ (f) Not listed

13. Pick the matrix below which is NOT diagonalizable:

(a) $\begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 4 \end{bmatrix}$

14. The exponential of the matrix $\begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}$ is

(a) $\begin{bmatrix} 0 & e^{-t} \\ e^t & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & -e^t \\ e^t & 0 \end{bmatrix}$

(c) $\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$

(d) $\begin{bmatrix} \cos t & -i\sin t \\ i\sin t & \cos t \end{bmatrix}$

(e) $\begin{bmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{bmatrix}$

(f) $\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$

Q1				ⓓ		
Q2				ⓓ		
Q3			ⓒ			
Q4				ⓓ		
Q5			ⓒ			
Q6					ⓔ	
Q7				ⓓ		

Q8					ⓔ	
Q9			ⓒ			
Q10			ⓒ			
Q11					ⓔ	
Q12					ⓔ	
Q13			ⓒ			
Q14			ⓒ			

Question 2. (20 points)
Find a solution to the 2^{nd} order differential equation

$$x''(t) + x(t) = 4|t| \cdot \sin(t), \qquad t \in \mathbb{R}$$

with initial conditions $x(0) = x'(0) = 0$.
Check that your solution is twice differentiable everywhere, including at $t = 0$.
Is it three times differentiable there? Why or why not?
Use this to write down all the (twice differentiable) solutions of the equation.
Hint: Consider the cases $t \geq 0$ and $t \leq 0$ separately and use them to assemble a solution on $\mathbb{R}$.

Two fundamental homogeneous solutions are $\cos t, \sin t$ for the entire real line.
For $t \geq 0$, we can solve by undetermined coefficients to get a particular solution $t \sin t - t^2 \cos t$.
For $t \leq 0$, we solve to get $t^2 \cos t - t \sin t$.
These particular solutions vanish at 0, along with their first and second derivatives. Splicing them up we get the function defined by $t \sin t - t^2 \cos t$ for $t \geq 0$ and by $t^2 \cos t - t \sin t$ for $t \leq 0$, which is is twice continuously differentiable at 0 and solves the equation.
All other solutions are obtained by adding linear combinations of $\sin t, \cos t$.
The third derivatives of the two half-line solutions also vanish at 0, so the function is in fact thrice differentiable. You can also see that from the equality $x''(t) = 4|t| \sin t - x(t)$: the right side is continuously differentiable.

Question 3. (10 points)
Find the solution with initial condition $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ for the ODE $\quad \dfrac{d\mathbf{x}}{dt}(t) = \begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix} \mathbf{x}(t)$.

The eigenvalues of the matrix are $4 \pm i$, with respective eigenvectors $\begin{bmatrix} 1 \\ \pm i \end{bmatrix}$. So the general (complex-valued) solution is $c_+ \exp((4+i)t) \begin{bmatrix} 1 \\ i \end{bmatrix} + c_- \exp((4-i)t) \begin{bmatrix} 1 \\ -i \end{bmatrix}$, with $c_+, c_- \in \mathbb{C}$. To match the initial condition at $t = 0$, we need $c_+ + c_- = 2$ and $c_+ - c_- = -2i$. So $c_\pm = 1 \mp i$ and the solution we want is

$$e^{4t} \left((1-i)(\cos t + i \sin t) \begin{bmatrix} 1 \\ i \end{bmatrix} + (1+i)(\cos t - i \sin t) \begin{bmatrix} 1 \\ -i \end{bmatrix} \right) = 2e^{4t} \begin{bmatrix} \cos t + \sin t \\ \cos t - \sin t \end{bmatrix}$$

Question 4. (15 points)
Find a particular solution for the following vector-valued ODE:

$$\mathbf{x}'(t) = \begin{bmatrix} -5 & 2 \\ -6 & 2 \end{bmatrix} \cdot \mathbf{x}(t) + \frac{1}{e^{2t}+1} \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

You may choose your method, but you must explain it briefly.
Help with integrals: $\int \frac{dt}{t^2+1} = \arctan(t) + C$

We use the eigenvector method. Diagonalizing the 2×2 matrix we find the eigenvalues (-1) and (-2) with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ respectively. Writing now $\mathbf{x}(t) = c_1(t)\mathbf{v}_1 + c_2(t)\mathbf{v}_2$, we note that $\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \mathbf{v}_1 + 2\mathbf{v}_2$ and the vector equation decouples into the two scalar equations for the coefficients of $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$c_1'(t) = -c_1(t) + \frac{1}{e^{2t}+1}$$
$$c_2'(t) = -2c_2(t) + \frac{2}{e^{2t}+1}$$

for which we find the particular solutions

$$c_1(t) = e^{-t} \int_0^t \frac{e^s ds}{e^{2s}+1}$$
$$c_2(t) = e^{-2t} \int_0^t \frac{2e^{2s} ds}{e^{2s}+1}$$

Both integrals can be done by substitution, the first with $u = e^s$ and the second with $v = e^{2s}$. The first gives $\arctan u - \pi/4$ and the second gives $\ln(v+1)$. We can omit the $-\pi/4$ if we don't insist that $c_1(0) = 0$, which is not required. So we can take for our answers

$$c_1(t) = e^{-t} \arctan(e^t)$$
$$c_2(t) = e^{-2t} \ln(e^{2t}+1)$$

and a particular solution is

$$c_1(t)\mathbf{v}_1 + c_2(t)\mathbf{v}_2 = \begin{bmatrix} e^{-t} \arctan(e^t) + 2e^{-2t} \ln(e^{2t}+1) \\ 2e^{-t} \arctan(e^t) + 3e^{-2t} \ln(e^{2t}+1) \end{bmatrix}$$

Question 5. (10 points)
Find all the numbers λ for which the differential equation $x''(t) = \lambda x(t)$ has *non-zero* solutions $x(t)$ which satisfy $x(0) = x(\pi) = 0$. For each such λ, write down all such solutions.
Suggestion: Write the general solution of the equation for a fixed λ, and adjust the constants to make $x(0), x(\pi)$ vanish. You may assume that λ is real, if it helps your calculation.

If $\lambda = 0$, then the general solution is a linear function $At + B$; vanishing at 0 and π will force it to vanish everywhere, so that will not work.
Fixing $\lambda \neq 0$, a pair of fundamental solutions are $\exp(\pm\mu t)$ where $\pm\mu$ are the two square roots of λ (for example, $\pm\mu = \pm i$ if $\lambda = -1$). Consider now the general solution $c_+e^{\mu t} + c_-e^{-\mu t}$. We need

$$c_+ + c_- = 0$$
$$c_+e^{\mu\pi} + c_-e^{-\mu\pi} = 0.$$

The determinant of the system matrix is $e^{-\mu\pi} - e^{\mu\pi}$, and its vanishing is needed for the system to have a non-zero solution. So we need $e^{2\mu\pi} = 1$. If $\mu = a + ib$, then $e^{\mu\pi} = e^{2a\pi}(\cos(2b\pi) + i\sin(2b\pi))$, and this equals 1 if and only if b is an integer and $a = 0$. Solving the system, we get the solutions $\sin(bt)$, b a non-zero integer (which we may take to be positive), for $\lambda = -b^2$. (So there are no solutions for non-real λ.)

Bonus Question. (5 points)
You can only get credit for this if you solved Q5 correctly.
(a) For any two twice-differentiable functions f, g which vanish at 0 and at π, show that

$$\int_0^\pi f''(t)g(t)dt = \int_0^\pi f(t)g''(t)dt.$$

(b) By using (a), or by direct computation, show that two solutions f, g as in Q5, but associated to two *different* values of λ are *orthogonal* in the sense that

$$\int_0^\pi f(t)g(t)dt = 0.$$

(a) Integration by parts gives

$$\int_0^\pi f''(t)g(t)dt = f'(t)g(t)\Big|_0^\pi - \int_0^\pi f'(t)g'(t)dt = f'(t)g(t)\Big|_0^\pi - f(t)g'(t)\Big|_0^\pi + \int_0^\pi f(t)g''(t)dt$$

and our conditions $f(0) = f(\pi) = g(0) = g(\pi) = 0$ make the difference between the integrals vanish.
(b) If f, g are solutions of the ODEs $f'' = \lambda f$ and $g'' = \mu g$ then we have that $\mu, \lambda \neq 0$ and

$$\int_0^\pi f(t)g(t)dt = \frac{1}{\lambda}\int_0^\pi f''(t)g(t)dt = \frac{1}{\mu}\int_0^\pi f(t)g''(t)dt$$

and $\lambda \neq \mu$ forces the vanishing of the integral.
Of course, knowing that (up to scale) $f(t) = \sin(mt), g(t) = \sin(nt)$ with distinct positive integers m, n allows us to compute directly

$$2\int_0^\pi \sin(mt)\sin(nt)dt = \int_0^\pi (\cos(mt - nt) - \cos(mt + nt))dt = \left(\frac{\sin(mt - nt)}{m - n} - \frac{\sin(mt + nt)}{m + n}\right)\Bigg|_0^\pi = 0$$