Mathematics 54
Final Exam, 16 December 2019
180 minutes, 90 points

NAME: ______________________ **ID:** ______________________

GSI: ____

INSTRUCTIONS:
Justify your answers, except when told otherwise.
All the work for a question should be on the respective sheet.

This is a CLOSED BOOK examination. Use of electronic devices (phones, tablets, etc) is NOT permitted. However, you may use a (two-sided) sheet of notes or formulas if you brought one with you.

Please turn in your finished examination **to your GSI** before leaving the room.

Credit: Q1:35; Q2:20; Q3:10; Q4:15; Q5:10. Bonus: 5

Question 1. (35 points) Select the correct answers, for 2.5 points each. No justification needed. Incorrect answers carry *no penalty* (but also no credit).
To receive credit, please record your choices by bubbling in the entries in the table on p.4. Mind how the answers are labeled (down columns).

1. When can we be certain that a system $A\mathbf{x} = \mathbf{b}$, with a 5×4 matrix A, is consistent?

(a) Always
(b) When $\mathbf{b}$ is in $\mathrm{Nul}(A)$
(c) When $\mathbf{b} \perp \mathrm{Nul}(A)$
(d) When $\mathbf{b} \perp \mathrm{LNul}(A)$
(e) When A has four pivots
(f) When $\mathrm{Nul}(A) = \{\mathbf{0}\}$

2. For which vector $\mathbf{b}$ below does the system $\begin{bmatrix} 2 & 4 \\ 4 & 6 \\ 3 & 4 \end{bmatrix} \mathbf{x} = \mathbf{b}$ have a solution?

(a) $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (f) $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

3. For general matrix A, which of the following must remain *unchanged* under row operations?
(i) The row space (ii) The column space (iii) The positions of the pivot columns

(a) (i) and (ii)
(b) (ii) and (iii)
(c) (i) and (iii)
(d) (i), (ii) and (iii)
(e) (ii) but not (i) or (iii)
(f) All of them can change

4. Which of the following collections of vectors in $\mathbb{R}^4$ are linearly dependent?

(a) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4$
(b) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_1$
(c) $\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_3 - \mathbf{e}_4$
(d) $\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_3 - \mathbf{e}_1$
(e) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$
(f) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_4$

5. Which of the matrices below have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 3 & 2 \\ 4 & 7 & 7 & 4 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 0 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 7 & 10 \end{bmatrix}; \quad D = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

(a) A, B and C but not D
(b) B and C but not A, D
(c) A and C but not B, D
(d) C and D but not A, B
(e) B, C and D but not A
(f) They all have rank 2

6. For a general $m \times n$ matrix A, the dimensions of $\mathrm{Col}(A)$ and of $\mathrm{Row}(A)$ agree *if and only if*

(a) A is symmetric
(b) A is square
(c) A is diagonalizable
(d) A is invertible
(e) They always agree!
(f) A is orthogonal

7. If A and B are square matrices of the same size, we can safely conclude that

(a) $AB = BA$
(b) $(A - B)(A + B) = A^2 - B^2$
(c) $AB^T = B^T A$
(d) $(AB)^T = B^T A^T$
(e) $(AB)^T = A^T B^T$
(f) None of the above.

8. If linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $T(\mathbf{e}_1 - \mathbf{e}_2) = \mathbf{e}_2 - \mathbf{e}_3$, $T(\mathbf{e}_2 - \mathbf{e}_3) = \mathbf{e}_3 - \mathbf{e}_1$ and $T(\mathbf{e}_3 - \mathbf{e}_1) = \mathbf{e}_1 - \mathbf{e}_2$, then we can be certain that

(a) T is invertible
(b) T is orthogonal
(c) $T(\mathbf{e}_1) = \mathbf{e}_2$
(d) $\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$ is in the range of T
(e) T has rank 2 or more
(f) T does not exist

9. The following is an eigenvalue of $A = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 1 & 5 \\ 0 & 0 & 7 \end{bmatrix}$:

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5 (f) (-1)

10. Let A be a 3×4 matrix. Which of the following statements about $A^T A$ *cannot* be true?

(a) It is square
(b) It is symmetric
(c) It is invertible
(d) It has rank 3
(e) It is diagonalizable over $\mathbf{R}$
(f) Its eigenvalues are ≥ 0

11. In which situation below can we be sure that the real $n \times n$ matrix A has *positive* determinant?

(a) A has positive entries
(b) There exists a matrix B with $AB = I_n$
(c) A has positive pivots
(d) A is diagonalizable
(e) All eigenvalues of A are positive real
(f) A is orthogonal

12. The least-squares solution to $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ is

(a) $\mathbf{x} = 0$ (b) $\mathbf{x} = 1$ (c) $\mathbf{x} = 2$ (d) $\mathbf{x} = 3$ (e) $\mathbf{x} = 4$ (f) Not listed

13. Pick the matrix below which is NOT diagonalizable:

(a) $\begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 4 \end{bmatrix}$

14. The exponential of the matrix $\begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}$ is

(a) $\begin{bmatrix} 0 & e^{-t} \\ e^t & 0 \end{bmatrix}$
(b) $\begin{bmatrix} 0 & -e^t \\ e^t & 0 \end{bmatrix}$
(c) $\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$
(d) $\begin{bmatrix} \cos t & -i \sin t \\ i \sin t & \cos t \end{bmatrix}$
(e) $\begin{bmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{bmatrix}$
(f) $\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$

Q1	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ	ⓕ
Q2	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ	ⓕ
Q3	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ	ⓕ
Q4	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ	ⓕ
Q5	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ	ⓕ
Q6	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ	ⓕ
Q7	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ	ⓕ

Q8	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ	ⓕ
Q9	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ	ⓕ
Q10	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ	ⓕ
Q11	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ	ⓕ
Q12	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ	ⓕ
Q13	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ	ⓕ
Q14	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ	ⓕ

Question 2. (20 points)
Find a solution to the 2nd order differential equation

$$x''(t) + x(t) = 4|t| \cdot \sin(t), \qquad t \in \mathbb{R}$$

with initial conditions $x(0) = x'(0) = 0$.
Check that your solution is twice differentiable everywhere, including at $t = 0$.
Is it three times differentiable there? Why or why not?
Use this to write down all the (twice differentiable) solutions of the equation.
Hint: Consider the cases $t \geq 0$ and $t \leq 0$ separately and use them to assemble a solution on $\mathbb{R}$.

Question 3. (10 points)

Find the solution with initial condition $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ for the ODE $\quad \dfrac{d\mathbf{x}}{dt}(t) = \begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix} \mathbf{x}(t).$

Question 4. (15 points)
Find a particular solution for the following vector-valued ODE:

$$\mathbf{x}'(t) = \begin{bmatrix} -5 & 2 \\ -6 & 2 \end{bmatrix} \cdot \mathbf{x}(t) + \frac{1}{e^{2t}+1} \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

You may choose your method, but you must explain it briefly.
Help with integrals: $\int \frac{dt}{t^2+1} = \arctan(t) + C$

Question 5. (10 points)
Find all the numbers λ for which the differential equation $x''(t) = \lambda x(t)$ has *non-zero* solutions $x(t)$ which satisfy $x(0) = x(\pi) = 0$. For each such λ, write down all such solutions.
Suggestion: Write the general solution of the equation for a fixed λ, and adjust the constants to make $x(0), x(\pi)$ vanish. You may assume that λ is real, if it helps your calculation.

Bonus Question. (5 points)
You can only get credit for this if you solved Q5 correctly.
(a) For any two twice-differentiable functions f, g which vanish at 0 and at π, show that

$$\int_0^\pi f''(t)g(t)dt = \int_0^\pi f(t)g''(t)dt.$$

(b) By using (a), or by direct computation, show that two solutions f, g as in Q5, but associated to two *different* values of λ are *orthogonal* in the sense that

$$\int_0^\pi f(t)g(t)dt = 0.$$

Cultural comment: This, plus Q5, show that the eigenfunctions of the second derivative operator, $f \mapsto f''$, form an orthogonal collection in the space of differentiable functions on $[0, \pi]$ vanishing at the endpoints. General theorems ensure that it is *complete*, so that any function above has a series expansion, convergent in mean square, in terms of the eigenfunctions you found in Q5.

THIS PAGE IS FOR ROUGH WORK (not graded)