EECS 16B Spring 2020 — Designing Information Devices and Systems II, UC Berkeley — Midterm 1

1. **CMOS Circuits (40 pts)**

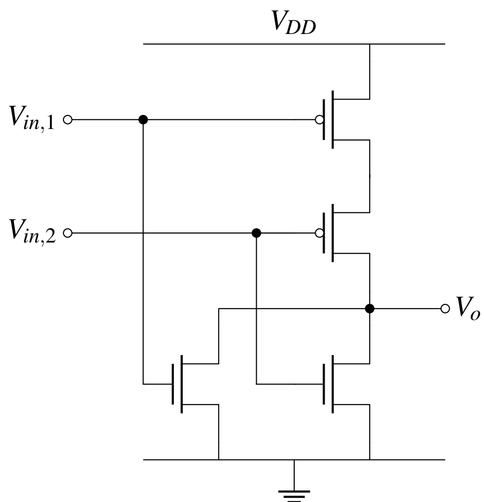


Figure 1: CMOS circuit

Consider the CMOS circuit of Figure 1. For each of the sets of $V_{in,1}$ and $V_{in,2}$ in the table below, fill in the corresponding voltage of the output V_o. You may assume that the threshold voltages for the transistors are $0 < V_{tn} < V_{DD}$ and $0 < |V_{tp}| < V_{DD}$.

$V_{in,1}$	$V_{in,2}$	V_o
0 V	0 V	
V_{DD}	0 V	
0 V	V_{DD}	
V_{DD}	V_{DD}	

Solution:

$V_{in,1}$	$V_{in,2}$	V_o
0 V	0 V	V_{DD}
V_{DD}	0 V	0
0 V	V_{DD}	0
V_{DD}	V_{DD}	0

2. Differential equations (40 pts)

Consider a certain radioactive isotope sample with an initial mass, M_0. It is observed that the mass of radioactive sample decays with time. The rate of mass decay at time t is proportional to the mass $M(t)$ at that moment. The constant of proportionality is the decay rate constant, $r > 0$. The rate of mass decay is given by

$$\frac{dM(t)}{dt} = -rM(t). \tag{1}$$

(a) (20 pts) **Solve the differential equation (1) to determine the mass of the sample at a given time.**

Solution: The solution of this differential equation with the initial condition, $M(0) = M_0$ is,

$$M = M_0 e^{-rt}.$$

$M(t) =$

(b) (20 pts) **Find the half life $t_{\frac{1}{2}}$ of the sample as a function of r. Note that half life is the time when half of the initial mass has decayed.** **Solution:** To find the half-life, $t_{\frac{1}{2}}$, we set, $M(t_{\frac{1}{2}}) = M_0/2$.

$$M_0/2 = M_0 e^{-rt_{\frac{1}{2}}}.$$

Solving, we find,

$$t_{\frac{1}{2}} = \frac{log_e(2)}{r}.$$

$t_{\frac{1}{2}} =$

3. Complex Numbers (60 pts)

(a) (20 pts) **Write the following numbers in polar form.** That is, write the numbers as $Ae^{j\theta}$ for some real numbers A and θ with $A \geq 0$ and $-\pi \leq \theta \leq \pi$.

i. $1+j$

Solution: $1+j = \sqrt{2}e^{j\pi/4}$ since $|1+j| = \sqrt{1^2+1^2}$ and the angle is $\text{atan2}(1,1) = \frac{\pi}{4}$

$1+j =$

ii. $\sqrt{j}$

Solution: We have

$$\begin{aligned}\sqrt{j} &= j^{1/2}\\ &= \left(e^{j\frac{\pi}{2}}\right)^{\frac{1}{2}}\\ &= e^{j\frac{\pi}{4}}\end{aligned}$$

This is not the only possible answer, since $j = e^{j\frac{5\pi}{2}}$ as well, so

$$\begin{aligned}\sqrt{j} &= j^{1/2}\\ &= \left(e^{j\frac{5\pi}{2}}\right)^{\frac{1}{2}}\\ &= e^{j\frac{5\pi}{4}}\end{aligned}$$

$\sqrt{j} =$

(b) (20 pts) **Write the following numbers in rectangular form.** That is, write each number as $a+bj$ where a and b are real numbers. HINT: $e^{j\theta} = \cos\theta + j\sin\theta$.

i. $3e^{j\frac{\pi}{3}}$

Solution: Using Euler's identity$3e^{j\frac{\pi}{3}} = 3\left(\cos\left(\frac{\pi}{3}\right) + j\sin\left(\frac{\pi}{3}\right)\right) = \frac{3}{2} + 3\frac{\sqrt{3}}{2}$

$3e^{j\frac{\pi}{3}} =$

ii. $-\sqrt{7}e^{\pi j}$

Solution: Note that $e^{j\pi} = -1$, so we have $-\sqrt{7}e^{\pi j} = \sqrt{7} + 0j$

$-\sqrt{7}e^{\pi j} =$

(c) (20 pts) **Prove the following identities.**

i. $\frac{1}{j} = -j$

Solution: We note that $\frac{1}{j} \times 1 = \frac{1}{j}\frac{j}{j} = \frac{j}{-1}$. The result follows.

Alternatively, use polar form to see that $j = e^{j\pi/4}$ and so $\frac{1}{j} = \frac{1}{e^{j\pi/4}} = e^{-j\pi/4} = -j$

ii. $\sin(2x) = 2\cos x\ \sin x$.

Solution:

Using the hint

$$\begin{aligned}
\sin(2x) &= \frac{e^{j(2x)} - e^{-j(2x)}}{2j} \\
&= \frac{\left(e^{jx}\right)^2 - \left(e^{-jx}\right)^2}{2j} \\
&= \frac{\left(e^{jx} + e^{-jx}\right)\left(e^{jx} - e^{-jx}\right)}{2j} \\
&= \left(e^{jx} + e^{-jx}\right)\left(\frac{e^{jx} - e^{-jx}}{2j}\right) \\
&= (2\cos x)(\sin x) \\
&= 2\cos x\ \sin x
\end{aligned}$$

4. Vector differential equations (60 pts)

For this problem, x satisfies the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = A\vec{x}; \quad A = \begin{bmatrix} \alpha & -\omega \\ \omega & \alpha \end{bmatrix},$$

where α and ω are real. Take initial condition $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(a) (30 pts) Using the following two eigenvectors of A:

$$v_1 = \begin{bmatrix} j \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -j \\ 1 \end{bmatrix},$$

determine a matrix T and a diagonal matrix D such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z} = Dz, \quad \text{where } \vec{z} = T\vec{x}.$$

You may use the fact that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Write your answers in the boxes below.

Solution:

Eigenvalues belonging to these eigenvectors are $\lambda_1 = \alpha + \omega j$ and $\lambda_2 = \alpha - \omega j$. For T:

$$T = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1} \tag{2}$$

$$= \begin{bmatrix} -j/2 & 1/2 \\ j/2 & 1/2 \end{bmatrix} \tag{3}$$

For D:

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} \alpha + \omega j & 0 \\ 0 & \alpha - \omega j \end{bmatrix} \tag{5}$$

$T =$
$D =$

(*extra space for (a)*)

(b) (15 pts) Solve for $\vec{z}$ as a function of t. Your solution may not include unknowns other than α and ω.

Solution:

A solution formula tells that

$$z_1 = z_1(0)e^{\lambda_1 t} \quad \text{and}$$
$$z_2 = z_2(0)e^{\lambda_2 t}$$

The initial condition $z(0)$ may be obtained by

$$\begin{aligned} z(0) &= Tx(0) \\ &= \begin{bmatrix} -j/2 \\ +j/2 \end{bmatrix} \end{aligned}$$

Hence

$$z = \begin{bmatrix} -\frac{1}{2}je^{(\alpha+\omega j)t} \\ +\frac{1}{2}je^{(\alpha-\omega j)t} \end{bmatrix}$$

(c) (15 pts) Solve for x_1, the first component of $\vec{x}$, as a function of t. If possible, state your solution without complex numbers.

Solution:
As $z = V^{-1}x$, where V is the matrix $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$,

$$\begin{aligned} x_1 &= jz_1 - jz_2 && (6) \\ &= \frac{1}{2}e^{(\alpha+\omega j)t} + \frac{1}{2}e^{(\alpha-\omega j)t} && (7) \\ &= e^{\alpha t}\cos(\omega t) && (8) \end{aligned}$$

5. Transient identification (40 pts)

For each of following scalar functions of time,

- If the function may describe a voltage transient response in one of the following circuit types we studied in class: RC, RL, RCRC, overdamped RLC, underdamped RLC; write all that apply.
- Otherwise, write "None."

Assume $V_0 \neq 0$, $V_1 \neq 0$, $V_2 \neq 0$ and $\omega > 0$.

(a) (10 pts) $v(t) = V_0 e^{\lambda t}$, $\lambda < 0$

Solution: RC, RL, and RCRC if the circuit has its resistors in parallel or series and capacitors in parallel or series. (Also Overdamped RLC, if the state stays in one eigenspace.)

(b) (10 pts) $v(t) = V_0 e^{jt}$

Solution: None.

(c) (10 pts) $v(t) = V_0 e^{\alpha t} \cos \omega t$, $\alpha < 0$

Solution: Underdamped RLC.

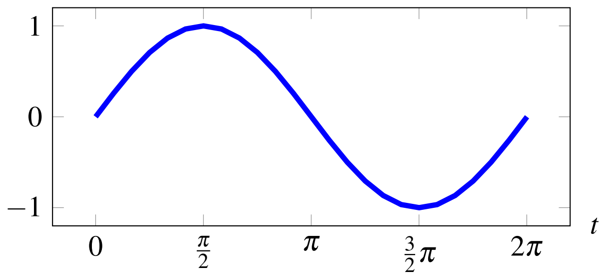
(d) (10 pts) $v(t) = V_1 e^{\alpha t} + V_2 e^{\beta t}$, $\alpha < 0$, $\beta < 0$

Solution: RCRC and overdamped RLC.

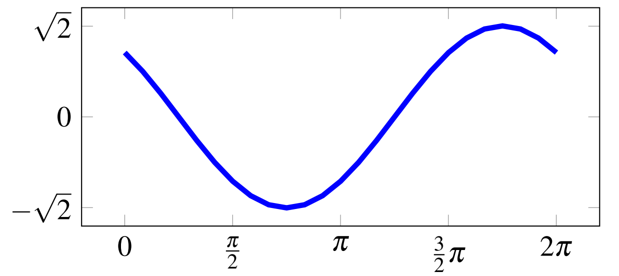
6. Phasors (60 pts)

Match each phasor with its corresponding time domain waveform that is shown below. For this problem $\omega = 1$.

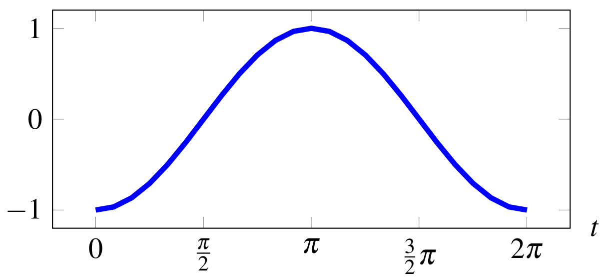
(1)



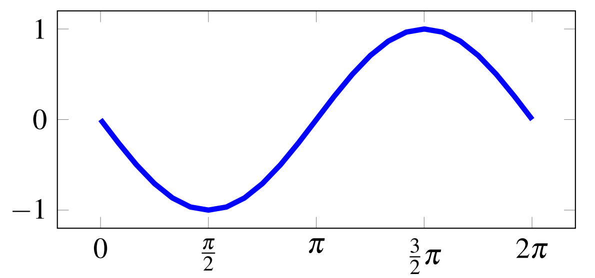
(4)



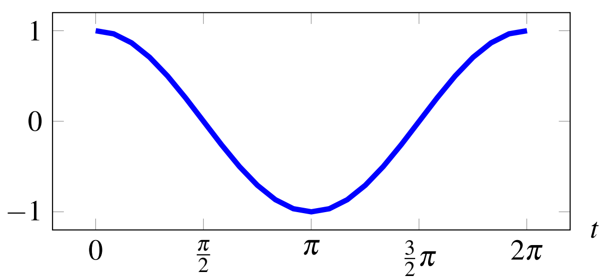
(2)



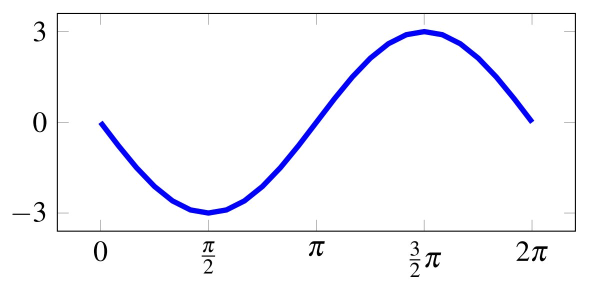
(5)



(3)



(6)

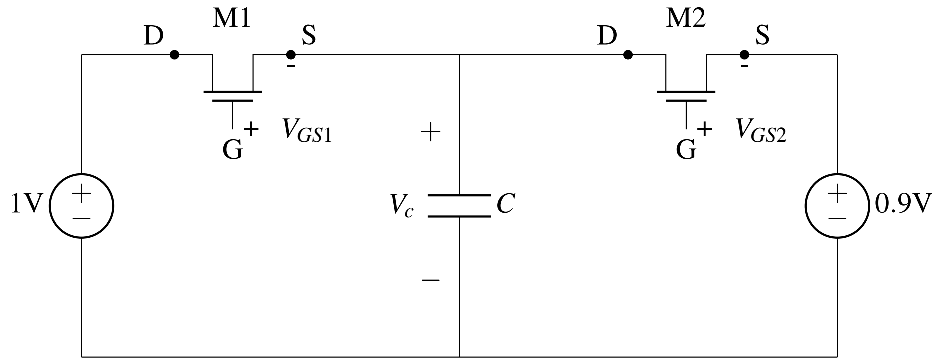


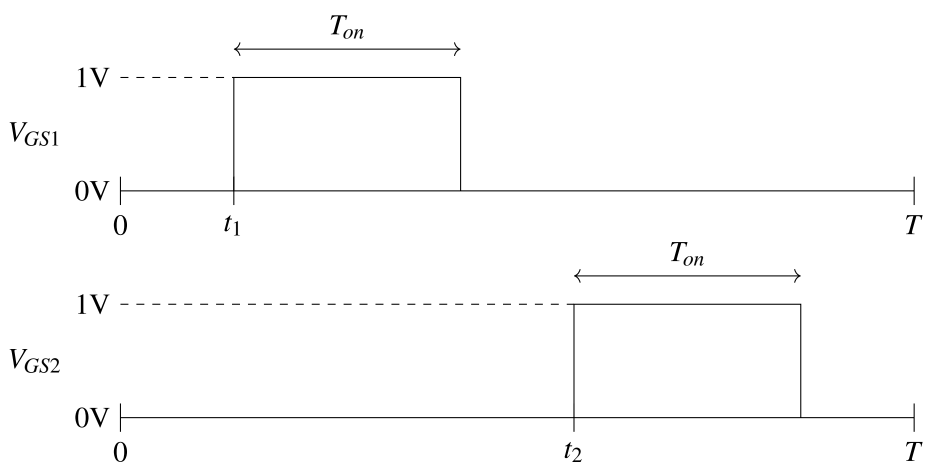
	Phasor	**Waveform**
(a)	$e^{j\frac{\pi}{2}}$	
(b)	$3e^{j\frac{\pi}{2}}$	
(c)	1	
(d)	$e^{-j\frac{\pi}{2}}$	
(e)	$e^{-j\pi}$	
(f)	$e^{j0}+e^{j\frac{\pi}{2}}$	

Solution:

	Phasor	**Waveform**
(a)	$e^{j\frac{\pi}{2}}$	5
(b)	$3e^{j\frac{\pi}{2}}$	6
(c)	1	3
(d)	$e^{-j\frac{\pi}{2}}$	1
(e)	$e^{-j\pi}$	2
(f)	$e^{j0}+e^{j\frac{\pi}{2}}$	4

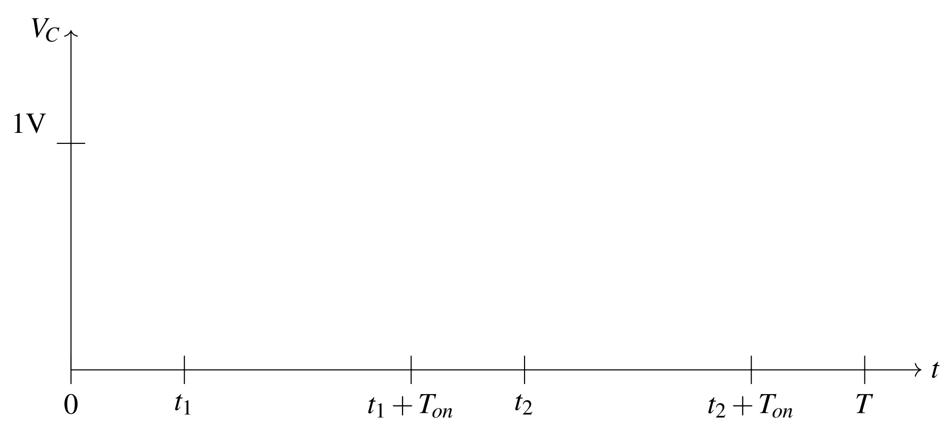
7. Energy (60 pts)





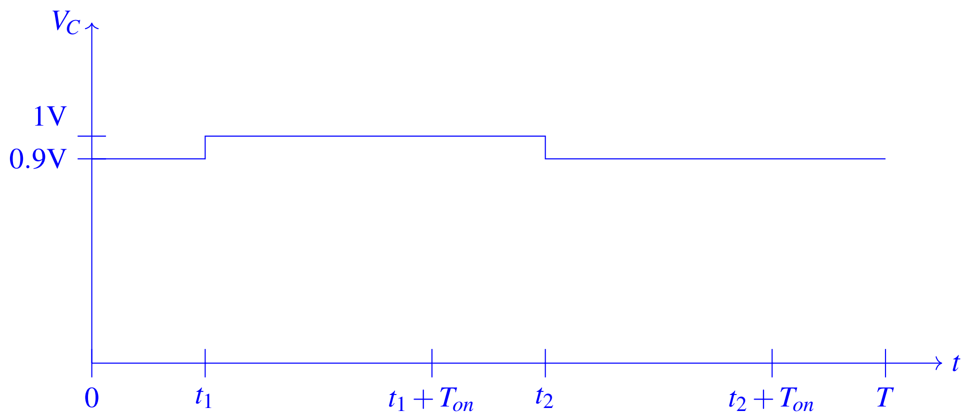
In the figure above, NMOS devices M1 and M2 have threshold voltages of 0.2V, negligible gate-source capacitances, and ON state resistances of value R. Capacitor C has capacitance such that $RC << T_{on}$. The associated timing diagram shows that V_{GS1} rises at t_1, and V_{GS2} rises at t_2.

(a) (20 pts) Take $V_C(0) = 0.9$V. On the axes below, sketch the waveform corresponding to $V_C(t)$ for $0 < t < T$. You may make approximations as informed by learnings in class to date.



Solution:

We have ignored the voltage transient in sketching the voltage due to the fact that $RC << T_{on}$.



(b) (20 pts) Determine how much charge flows out of the 1.0V source over the period from 0 to T. You may make reasonable approximations to simplify the analysis, as done in the making of the sketch of part (a).

Solution: At $t = t_1$, the capacitor charges from 0.9V to 1V, and this charge is supplied from the source. This is the only time period where the source supplies charge. In order for the voltage on the capacitor to increase by $\Delta V = 0.1$V, the amount of charge required is:

$$Q_{supplied} = C\Delta V \rightarrow Q_{supplied} = C(0.1V)$$

(c) (20 pts) Determine how much energy is dissipated in the circuit over the period from 0 to T. Again, you may make reasonable approximations to simplify your analysis.

Solution: The energy dissipated in the circuit is equal to the energy supplied by the 1V source E_{1V} minus the energy delivered to the 0.9V source $E_{0.9V}$. From the previous part, we know that the energy supplied by the source is:

$$E_{1V} = Q_{supplied}(1V) = C(0.1V)(1V) = C(0.1)J$$

The charge delivered to the 0.9V source must be the same as the charge supplied by the source, or else the voltage on the capacitor would go to plus/minus infinity as time goes towards infinity. We can also explicitly calculate the charge delivered to the 0.9V source as:

$$Q_{delivered} = C(1V - 0.9V) = C(0.1V)$$

Therefore, the energy delivered to the 0.9V source is:

$$E_{0.9} = Q_{delivered}(0.9V) = C(0.1V)(0.9V) = C(0.09)J$$

Finally the energy dissipated in the circuit is:

$$E_{1V} - E_{0.9} = C(0.1)J - C(0.09)J = C(0.01)J$$

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