Midterm 1 Solutions

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February 24, 2020

Imagine an electron trapped in a potential, something like a box. In this particular system, the lowest two energy states are much lower in energy than the others, and so, to a very good approximation, we can take the system's state space to be two-dimensional. Under this approximation, we can use the orthonormal Hamiltonian eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$ as a basis for the state space. We've learned that the energy eigenvalues for these states are $E_1 = 0$ and $E_2 = 4$.

In addition to this information about the energy, we have discovered that, in this two-dimensional state space, the vectors $i|\phi_1\rangle + 2|\phi_2\rangle$ and $2|\phi_1\rangle + i|\phi_2\rangle$ are eigenvectors of the position operator, with eigenvalues -1 and 1, respectively.

1) (10 points): Which of the following two vectors is not a valid system state vector? Explain.

$$|u\rangle = \frac{1}{\sqrt{2}}|\phi_2\rangle - \frac{i}{\sqrt{2}}|\phi_1\rangle \qquad\qquad |v\rangle = |\phi_1\rangle + i|\phi_2\rangle$$

From the first postulate, a valid system state vector is normalized. From the problem statement we know that the Hamiltonian eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthonormal. Therefore $\langle\phi_1|\,\phi_1\rangle = \langle\phi_2|\,\phi_2\rangle = 1$ and $\langle\phi_1|\,\phi_2\rangle = \langle\phi_2|\,\phi_1\rangle = 0$.

$$\begin{aligned}
\langle u|\,u\rangle &= \left(\left(\frac{1}{\sqrt{2}}\right)^*\langle\phi_2| - \left(\frac{i}{\sqrt{2}}\right)^*\langle\phi_1|\right)\left(\frac{1}{\sqrt{2}}|\phi_2\rangle - \frac{i}{\sqrt{2}}|\phi_1\rangle\right)\\
&= \frac{1}{2}\langle\phi_2|\,\phi_2\rangle - \frac{i}{2}\langle\phi_2|\,\phi_1\rangle + \frac{i}{2}\langle\phi_1|\,\phi_2\rangle + \frac{1}{2}\langle\phi_1|\,\phi_1\rangle\\
&= \frac{1}{2} + \frac{1}{2}\\
&= 1
\end{aligned}$$

$$\begin{aligned}
\langle v|\,v\rangle &= (\langle\phi_1| - i\langle\phi_2|)(|\phi_1\rangle + i|\phi_2\rangle)\\
&= \langle\phi_1|\,\phi_1\rangle + i\langle\phi_1|\,\phi_2\rangle - i\langle\phi_2|\,\phi_1\rangle + \langle\phi_2|\,\phi_2\rangle\\
&= 1 + 1\\
&= 2\\
&\neq 1
\end{aligned}$$

Therefore $|v\rangle$ is not a valid system state vector because it is not normalized, while $|u\rangle$ is.

2) (15 points) Show that it is not possible to know both the energy and position of the electron in this system at the same time. Hint: knowing the form of the potential is not necessary here.

From the problem statement we know that energy and position have different eigenstates. The Hamiltonian, which is the total energy operator, eigenstates are $|\phi_1\rangle$ and $|\phi_2\rangle$ while the position eigenstates are superpositions of $|\phi_1\rangle$ and $|\phi_2\rangle$. Therefore we can't know both the energy and the position of the electron.

Similar arguments can be made using the commutator $[\hat{H}, \hat{x}]$. See the week 3 discussion notes for evaluation of this commutator with the relevant Hamiltonian.

3) (15 points) Imagine preparing the electron in the position eigenstate with position -1 and then measuring its energy. If we repeated this process a large number of times, what would the average of these energy measurements be?

Because we are preparing the position of the electron before each measurement, the state we must consider is the position eigenstate with position -1. For use in an expectation value, this eigenstate must first be normalized.

$$\begin{aligned}
1 &= \langle x_{-1}| x_{-1}\rangle \\
&= a^* a \left(-i \langle\phi_1| + 2 \langle\phi_2|\right)\left(i |\phi_1\rangle + 2 |\phi_2\rangle\right) \\
&= a^* a \left(\langle\phi_1| \phi_1\rangle - 2i \langle\phi_1| \phi_2\rangle + 2i \langle\phi_2| \phi_1\rangle + 4 \langle\phi_2| \phi_2\rangle\right) \\
&= a^* a (1+4) \\
\Rightarrow a &= \frac{1}{\sqrt{5}} \\
\Rightarrow |x_{-1}\rangle &= \frac{1}{\sqrt{5}}\left(i |\phi_1\rangle + 2 |\phi_2\rangle\right)
\end{aligned}$$

$$\begin{aligned}
\langle E\rangle &= \langle x_{-1}| \hat{H} |x_{-1}\rangle \\
&= \frac{1}{\sqrt{5}}(-i \langle\phi_1| + 2 \langle\phi_2|)\hat{H}\frac{1}{\sqrt{5}}(i |\phi_1\rangle + 2 |\phi_2\rangle) \\
&= \frac{1}{5}(-i^2 \langle\phi_1| \hat{H} |\phi_1\rangle + 2^2 \langle\phi_2| \hat{H} |\phi_2\rangle) \\
&= \frac{1}{5}(-i^2(0) + 2^2(4)) \\
&= \frac{16}{5}
\end{aligned}$$

4) (25 points) if at time $t = 0$ we measure the electron's position and observe the value -1, what is the probability of observing a position value of 1 in a second measurement made at $t = \pi/8$? For simplicity, you may set $\hbar = 1$.

At $t = 0$ our initial state is

$$|\psi(0)\rangle = |x_{-1}\rangle = \frac{1}{\sqrt{5}}(i\,|\phi_1\rangle + 2\,|\phi_2\rangle)$$

from the state collapse postulate. Then using the time-dependent Schrodinger equation we can express the time-dependence of our initial state as

$$|\psi(t)\rangle = \frac{1}{\sqrt{5}}(ie^{i(0)t/\hbar}\,|\phi_1\rangle + 2e^{i(4)t/\hbar}\,|\phi_2\rangle) = \frac{1}{\sqrt{5}}(i\,|\phi_1\rangle + 2e^{i4t}\,|\phi_2\rangle)$$

Now consider how the probability of observing a position value of 1 changes with time.

$$\begin{aligned}
prob(x = 1, t) &= |\,\langle x_1|\,\psi(t)\rangle\,|^2 \\
&= |(\frac{1}{\sqrt{5}}(2\,\langle\phi_1| - i\,\langle\phi_2|))(\frac{1}{\sqrt{5}}(i\,|\phi_1\rangle + 2e^{i4t}\,|\phi_2\rangle))|^2 \\
&= \frac{1}{25}|((2\,\langle\phi_1| - i\,\langle\phi_2|))((i\,|\phi_1\rangle + 2e^{i4t}\,|\phi_2\rangle))|^2 \\
&= \frac{1}{25}|(2i - 2ie^{i4t})|^2 \\
&= \frac{4}{25}|(1 - e^{i4t})|^2 \\
&= \frac{4}{25}\big((1 - e^{i4t})(1 - e^{-i4t})\big) \\
&= \frac{4}{25}\big(1 - e^{-i4t} - e^{i4t} + 1\big) \\
&= \frac{4}{25}\big(2 - (e^{-i4t} + e^{i4t})\big) \\
&= \frac{4}{25}\big(2 - 2\cos(4t)\big) \\
&= \frac{8}{25}(1 - \cos(4t))
\end{aligned}$$

$$prob(x = 1, t = \pi/8) = \frac{8}{25}(1 - \cos(4 \times \pi/8)) = \frac{8}{25}(1 - \cos(\pi/2)) = \frac{8}{25}(1 - 0) = \frac{8}{25}$$

5) (20 points) If the momentum operator's eigenvalues are 3 and 4, prove that the two corresponding momentum eigenvectors must be orthogonal.

Based on the problem statement we can write $\hat{p}\,|p_3\rangle = 3\,|p_3\rangle$ and $\hat{p}\,|p_4\rangle = 4\,|p_4\rangle$. Since the momentum operator is Hermitian, $\hat{p}^\dagger = \hat{p}$. Therefore we can write:

$$\begin{aligned}
\langle p_3|\,\hat{p}\,|p_4\rangle &= \langle p_3|\,\hat{p}^\dagger\,|p_4\rangle \\
0 &= \langle p_3|\,\hat{p}^\dagger\,|p_4\rangle - \langle p_3|\,\hat{p}\,|p_4\rangle \\
&= \left(\langle p_3|\,\hat{p}^\dagger\right)|p_4\rangle - \langle p_3|\,(\hat{p}\,|p_4\rangle) \\
&= (3^*\,\langle p_3|)\,|p_4\rangle - \langle p_3|\,(4\,|p_4\rangle) \\
&= 3\,\langle p_3|\,p_4\rangle - 4\,\langle p_3|\,p_4\rangle \\
&= (3-4)\,\langle p_3|\,p_4\rangle \\
&= -\,\langle p_3|\,p_4\rangle
\end{aligned}$$

This expression only holds if $\langle p_3|\,p_4\rangle = 0$, which is the orthogonality condition for the momentum eigenvectors $|p_3\rangle$ and $|p_4\rangle$.

6) (15 points) Briefly explain why, in the double slit experiment, the interference pattern goes away when you measure which slit the electron passes through. A couple sentences is enough.

The interference pattern measured at the detector of the double slit experiment arises from the electron's wave function containing two components in superposition. These two pieces pass through the two different slits and then interfere with each other (much as two water waves would in a pool). When you make a measurement at the slits you learn that the electron passes through one slit or the other, and the the collapse postulate causes the wave function to become just one of the two original pieces (whichever corresponds to what you learned in the measurement). Now, with just one component of the wave function remaining, the interference effect no longer occurs.