

S20 PHYSICS 7B: Bordel MT 1 Solutions

February 24, 2020

1 Problem 1

(a)

By the ideal gas law:

$$T_f = \frac{P_f V_f}{nR} = \frac{1.1 P_i V_i}{nR} = 1.1 T_i. \quad (1.1)$$

(b)

Our equation for linear expansion says:

$$\frac{V_f}{V_i} = 1.1 = (1 + \alpha \Delta T)^3 \sim 1 + 3\alpha \Delta T = 1 + 0.3\alpha T_i. \quad (1.2)$$

We hence find:

$$\alpha = \frac{1}{3T_i}. \quad (1.3)$$

(c)

The equipartition theorem says the average kinetic energy of a single molecule is:

$$\langle E \rangle = \frac{3}{2} kT = 3kT = \frac{1}{2} m \langle v^2 \rangle \quad \implies \quad v_{\text{rms}} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3kT}{m}}. \quad (1.4)$$

Then the ratio of RMS velocities is:

$$\frac{v_{f,\text{rms}}}{v_{i,\text{rms}}} = \sqrt{\frac{T_f}{T_i}} = \sqrt{1.1} > 1. \quad (1.5)$$

The result is greater than 1, because the temperature after the expansion is larger, implying the average kinetic energy of the particles is larger.

(d)

As a monatomic gas, neon has no rotational degrees of freedom, so $d = 3$. For water molecules with no vibrational modes, $d = 6$, corresponding to 3 translational and 3 rotational degrees of freedom. Then the equipartition theorem says the average energy of a n moles of water vapor is:

$$\Delta E_W = 3nR(T_f - T_i) = \frac{3}{10} nRT_i, \quad (1.6)$$

while the change of internal energy of neon gas is

$$\Delta E_N = \frac{3}{2} nR(T_f - T_i) = \frac{3}{20} nRT_i < \Delta E_W. \quad (1.7)$$

We see that the change in energy is greater for the neon gas – this is because the heat capacity at constant pressure C_P is smaller for neon gas, due to the fact that it does not have rotational degrees of freedom.

2 Problem 2

(a)

By the ideal gas law:

$$P_A T_A = C = \frac{P_A^2 V_A}{nR}, \quad (2.1)$$

where C is a constant. We hence have

$$P_A^2 V_A = P_B^2 V_B = (2P_A)^2 V_B = 4P_A^2 V_B. \quad (2.2)$$

We conclude $V_B = \frac{1}{4}V_A$.

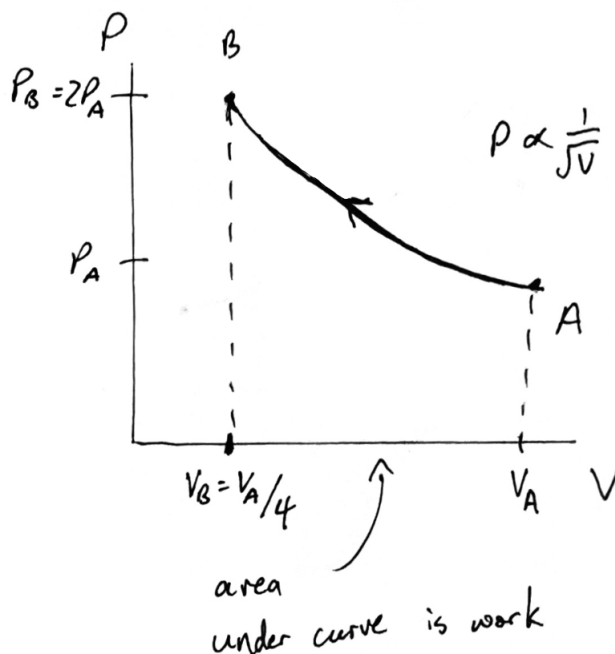


Figure 1: Diagram for 2a, 2b

(b)

The work is computed as:

$$W = \int_{V_A}^{V_B} P dV = \int_{V_A}^{V_A/4} \sqrt{\frac{nRC}{V}} dV = 2\sqrt{nRC}(\sqrt{V_A/4} - \sqrt{V_A}) = -\sqrt{nRCV_A}. \quad (2.3)$$

Then using $nRC = P_A^2 V_A$, we find

$$W = -P_A V_A. \quad (2.4)$$

The work is negative because the volume of the gas decreased. Graphically, the work is given by the area underneath the curve on a PV -diagram.

(c)

The heat is $Q = \Delta E_{AB} + W$. From the first part, we know:

$$P_B^2 = P_A^2 \frac{V_A}{V_B} = 4P_A^2, \quad (2.5)$$

and hence the ideal gas law says

$$T_B = \frac{P_B V_B}{nR} = \frac{P_A V_A}{2nR}. \quad (2.6)$$

Then the equipartition theorem with $d = 3$ for a monatomic ideal gas gives

$$\Delta E_{AB} = \frac{3}{2}nR(T_B - T_A) = -\frac{3}{4}P_A V_A. \quad (2.7)$$

We hence find

$$Q = -\frac{7}{4}P_A V_A \quad (2.8)$$

(d)

The change in entropy is:

$$\Delta S = \int \frac{dQ}{T} = \int \frac{dE + dW}{T} = \frac{3}{2}nR \int_{T_A}^{T_B} \frac{dT}{T} + \int_{V_A}^{V_B} \frac{P}{T} dV. \quad (2.9)$$

The first integral can be evaluated as

$$\frac{3}{2}nR \int_{T_A}^{T_B} \frac{dT}{T} = \frac{3}{2}nR \ln \frac{T_B}{T_A} = \frac{3}{2}nR \ln \frac{P_B V_B}{P_A V_A} = \frac{3}{2}nR \ln \frac{\frac{1}{2}P_A V_A}{P_A V_A} = -\frac{3}{2}nR \ln 2. \quad (2.10)$$

The second integral can be evaluated by noting:

$$\int_{V_A}^{V_B} \frac{P}{T} dV = \int_{V_A}^{V_B} \frac{nRT}{TV} dV = nR \int_{V_A}^{V_B} \frac{dV}{V} = nR \ln \frac{V_B}{V_A} = -nR \ln 4 = -2nR \ln 2. \quad (2.11)$$

We hence conclude $\Delta S = -\frac{7}{2}nR \ln 2$. This result does not violate the 2nd Law, because the environment entropy increases by at least the same amount so the total entropy of the process is non-negative.

3 Problem 3

(a)

Taking room temperature to be around 30 degrees Celsius, we follow the line at 1 atm, and we see that at equilibrium, the material will be in the vapor phase.

(b)

The plot has 3 parts, labeled by a, b, c, d . In a , the material is solid and warming up due to exposure to a higher temperature. At -78.4 degrees Celsius, the material begins to undergo a phase transformation, as depicted by b . Here, the temperature stays constant over time as the energy from the environment causes the material to change phase. After the phase change, the material is in the vapor stage, depicted by c . The vapor now continues to increase in temperature, until it reaches room temperature, which we have modeled as 30 degrees Celsius. In d , the vapor is now in equilibrium and stays at a constant temperature.

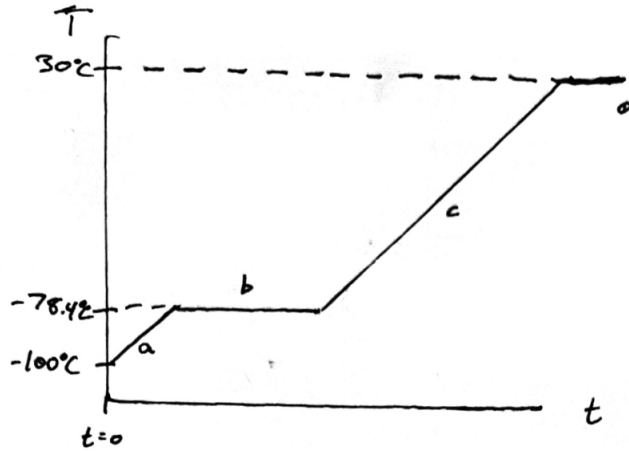


Figure 2: Plot for 3b.

(c)

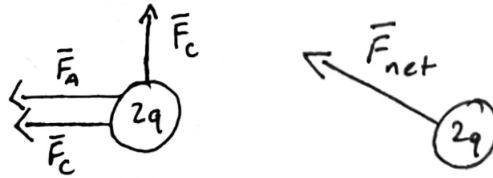


Figure 3: Diagram for 3c.

(d)

We superimpose the electric field due to the three point charges at A , B , and C at point O . The fields are:

- $\vec{E}_A(O) = -\frac{kq}{d^2}\hat{i}$
- $\vec{E}_B(O) = \frac{kq}{d^2}\hat{j}$
- $\vec{E}_C(O) = -\frac{kq}{d^2}\hat{i}$.

The total field at O from the three point charges is therefore:

$$\vec{E}_{ABC}(O) = \vec{E}_A(O) + \vec{E}_B(O) + \vec{E}_C(O) = -\frac{2kq}{d^2}\hat{i} + \frac{kq}{d^2}\hat{j}. \quad (3.1)$$

The force on a $2q$ charge at O is therefore:

$$\vec{F} = 2q\vec{E}_{ABC}(O) = -\frac{4kq^2}{d^2}\hat{i} + \frac{2kq^2}{d^2}\hat{j}. \quad (3.2)$$

4 Problem 4

(a)

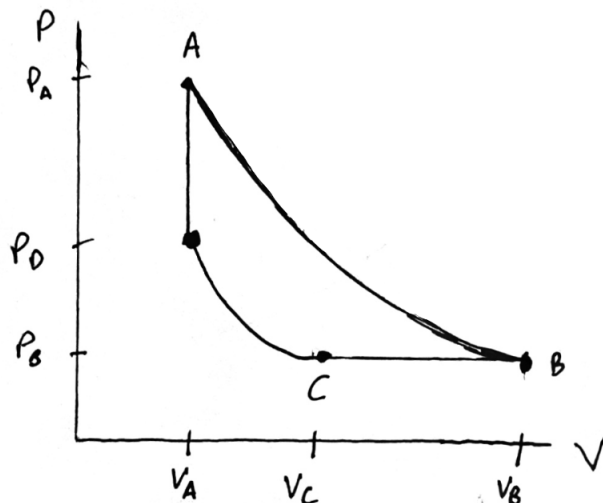


Figure 4: Diagram for 4a.

By the ideal gas law:

$$P_A V_A = nRT_A \implies T_A = \frac{P_A V_A}{nR}. \quad (4.1)$$

The adiabatic index is $\gamma = \frac{d+2}{d}$ and here $d = 2$, so $\gamma = 2$. Then the result of the expansion $A \rightarrow B$ gives:

$$P_A V_A^2 = \text{constant} = P_B V_B^2 = P_B (4V_A)^2 = 16P_B V_A^2 \implies P_B = \frac{1}{16} P_A, \quad (4.2)$$

and hence

$$T_B = \frac{P_B V_B}{nR} = \frac{P_A V_A}{4nR}. \quad (4.3)$$

The result of the isobaric compression $B \rightarrow C$ can be computed from the ideal gas law directly:

$$P_C V_C = 2P_B V_A = \frac{1}{8} P_A V_A \implies T_C = \frac{P_A V_A}{8nR}. \quad (4.4)$$

Finally, the adiabatic compression $C \rightarrow D$ implies

$$P_C V_C^2 = \frac{1}{16} P_A (2V_A)^2 = \frac{1}{4} P_A V_A^2 = P_D V_D^2 \implies P_D = \frac{1}{4} P_A. \quad (4.5)$$

Hence the ideal gas law gives

$$T_D = \frac{P_D V_D}{nR} = \frac{P_A V_A}{4nR}. \quad (4.6)$$

(b)

There are two equivalent approaches; the first is to explicitly compute the work in each processes as follows:

- $A \rightarrow B$: $W = -\Delta E_{AB} = -\frac{d}{2}nR(T_B - T_A) = \frac{3}{4}P_A V_A$
- $B \rightarrow C$: $W = P_B(2V_A - 4V_A) = -\frac{1}{8}P_A V_A$
- $C \rightarrow D$: $W = -\Delta E_{CD} = -nR(T_D - T_C) = -\frac{1}{8}P_A V_A$
- $D \rightarrow A$: $W = 0$

Summing everything, we find $W_{\text{net}} = \frac{1}{2}P_A V_A$. The second approach is to use the fact that the energy is a state function, and hence $\Delta E = 0$ for the entire cycle. This then implies $W_{\text{net}} = Q_{\text{net}}$. The heat in each process is:

- $A \rightarrow B$: $Q = 0$
- $B \rightarrow C$: $Q = nC_P(T_C - T_B) = -\frac{1}{4}P_A V_A$
- $C \rightarrow D$: $Q = 0$
- $D \rightarrow C$: $Q = nC_V(T_A - T_D) = \frac{3}{4}P_A V_A$

We hence find $W_{\text{net}} = Q_{\text{net}} = \frac{1}{2}P_A V_A$.

(c)

The efficiency is:

$$\eta = \frac{W_{\text{net}}}{Q_{\text{in}}} = \frac{\frac{1}{2}P_A V_A}{\frac{3}{4}P_A V_A} = \frac{2}{3} \quad (4.7)$$

(d)

The Carnot efficiency is

$$\eta_C = 1 - \frac{T_{\text{cold}}}{T_{\text{hot}}} = 1 - \frac{T_C}{T_A} = 1 - \frac{1}{8} = \frac{7}{8}. \quad (4.8)$$

We see $\eta_C \geq \eta$, as expected.

5 Problem 5

(a)

The heat flow equation is:

$$\frac{dQ}{dt} = -kA(x)\frac{dT}{dx}, \quad (5.1)$$

where we set $x = 0$ to the left and $x = L$ to the right. The cross-sectional area $A(x)$ is $A(x) = \pi [R_1 + \frac{x}{L}(R_2 - R_1)]^2$. Because the rod is perfectly insulated, all heat that enters the rod leaves the other end. Given a constant flow rate C :

$$-kA(x)\frac{dT}{dx} = C \quad \implies \quad -\frac{k}{C}dT = \frac{1}{A(x)}dx \quad (5.2)$$

We can integrate each side:

$$-\frac{k}{C} \int_{T_F}^{T_B} dT = \int_0^L \frac{dx}{\pi [R_1 + \frac{x}{L}(R_2 - R_1)]^2}, \quad (5.3)$$

Choosing $u = R_1 + \frac{x}{L}(R_2 - R_1)$ with $du = \frac{R_2 - R_1}{L} dx$, we obtain:

$$-\frac{k}{C}(T_B - T_F) = \frac{L}{\pi(R_2 - R_1)} \int_{R_1}^{R_2} \frac{du}{u^2} = -\frac{L}{\pi(R_2 - R_1)} \frac{R_1 - R_2}{R_1 R_2} = \frac{L}{\pi R_1 R_2}. \quad (5.4)$$

Then solving for the flow rate C :

$$C = \frac{k\pi R_1 R_2 (T_F - T_B)}{L}. \quad (5.5)$$

Because $T_B > T_F$, the heat flow is in the negative x direction, corresponding to the heat flowing from the boiling water to the ice, as expected.

(b)

In the left container, we have a mass m of water at 273 K, and in the right container, we have a mass $m/2$ of water at 373 K. Then the calorimetry equation is

$$mC_w(T_f - T_F) + \frac{1}{2}mC_w(T_f - T_B) = 0, \quad (5.6)$$

where T_f is the final temperature of the water. Solving, we find:

$$T_f = \frac{2T_F + T_B}{3} \quad (5.7)$$

(c)

The change in entropy of the water is determined by the integral:

$$\Delta S = \int \frac{dQ}{T} = mC_w \int_{T_F}^{T_f} \frac{dT}{T} + \frac{1}{2}mC_w \int_{T_B}^{T_f} \frac{dT}{T} = mC_w \left(\log \frac{T_f}{T_F} + \frac{1}{2} \log \frac{T_f}{T_B} \right). \quad (5.8)$$

The container is insulated, so the surroundings receive no heat, and hence there is no change in entropy of the surroundings. The change above is therefore the total change in entropy.