EECS 16A Designing Information Devices and Systems I
Spring 2019 Midterm 1

1. What is one of your hobbies? (1 point)

2. Tell us about something that makes you happy. (1 point)

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

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3. Quadcopter (12 points)

Thanks to the amazing linear algebra and circuits skills you learned in EE 16A, you have been hired by a local startup, BearQuad, a hot new delivery service that brings food to customers across the UC Berkeley campus via quadcopter. You have been tasked with developing a way to determine the thrust force each propeller must produce for the quadcopter to hover based on physical constraints on the quadcopter.

You decide to go with the standard quadcopter design with four propellers, each spinning as shown in Figure 3.1. Each propeller is ℓ away from the center of mass of the quadcopter.

Next, you formulate some equations to describe the dynamics of the quadcopter, which are listed in Figure 3.2. For the quadcopter to hover, the sum of the thrust forces (f_1, f_2, f_3, and f_4) must equal the weight of the quadcopter and its payload (f_W). Furthermore, for the quadcopter to reach a certain orientation, the propellers must achieve specific torques (n_x, n_y, and n_z) about each of the x-, y-, and z-axes, and these torques are functions of the propeller forces. In the n_z equation, k is an experimentally-determined constant, and the sign in front of k depends on the direction the propeller is spinning.

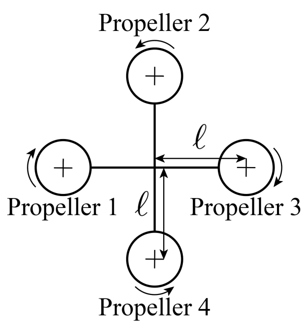

Figure 3.1: Diagram of the quadcopter design.

$$
\begin{aligned}
f_W &= f_1 + f_2 + f_3 + f_4 \\
n_x &= -\ell f_1 + \ell f_3 \\
n_y &= \ell f_2 - \ell f_4 \\
n_z &= -k f_1 + k f_2 - k f_3 + k f_4
\end{aligned}
$$

Figure 3.2: Quadcopter dynamics equations.

(a) (2 points) Using the equations you determined about the quadcopter dynamics and in terms of l and k, find a matrix $\mathbf{A}$ such that

$$
\begin{bmatrix} f_W \\ n_x \\ n_y \\ n_z \end{bmatrix} = \mathbf{A} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}
$$

Solution: Assemble the matrix $\mathbf{A}$ from the system of equations given in the problem statement:

$$
\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -\ell & 0 & \ell & 0 \\ 0 & \ell & 0 & -\ell \\ -k & k & -k & k \end{bmatrix}
$$

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(b) (4 points) Regardless of your answer in the previous part, assume that after you measure the constants ℓ and k, you get the following for matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 10 & 10 & 10 & 10 \\ -2 & 2 & 2 & -2 \\ 2 & 2 & -2 & -2 \\ -0.1 & 0.1 & -0.1 & 0.1 \end{bmatrix}$$

The matrix $\mathbf{A}$ converts individual thrust forces to f_W and the torques. However, as explained in the preamble to this problem, your task is to get the *individual forces from f_W and the torques*. Let a new matrix $\mathbf{B}$ convert the total force and torques to the individual forces such that

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \mathbf{B} \begin{bmatrix} f_W \\ n_x \\ n_y \\ n_z \end{bmatrix}$$

Does this matrix B exist? Justify using Gaussian Elimination.

Solution: To reverse the conversion, B must be the inverse of A. We can show that A is invertible by showing its columns are linearly independent (Invertible Matrix Theorem). We do this by performing Gaussian Elimination:

$$\begin{bmatrix} 10 & 10 & 10 & 10 \\ -2 & 2 & 2 & -2 \\ 2 & 2 & -2 & -2 \\ -0.1 & 0.1 & -0.1 & 0.1 \end{bmatrix} \xrightarrow{\substack{R_1 \leftarrow R_1/10 \\ R_2 \leftarrow R_2/-2 \\ R_3 \leftarrow R_3/2 \\ R_4 \leftarrow R_4/-0.1}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{\substack{R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1 \\ R_4 \leftarrow R_4 - R_1}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2/-2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 + 2R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 2 & -2 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3/-2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$$\xrightarrow{R_4 \leftarrow (R_4 - 3R_2)/-4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{R_1 \leftarrow R_1 - R_4 \\ R_3 \leftarrow R_3 - R_4}} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{R_1 \leftarrow R_1 - R_3 \\ R_2 \leftarrow R_2 - R_3}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The row reduced matrix is the identity matrix, so there are the same number of pivots as columns (4). Thus, the matrix's columns are linearly independent, A is invertible, and B exists.

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(c) (6 points) You show your design to your boss. Wanting to be different from other quadcopter companies, your boss suggests you switch the spinning directions for propellers 3 and 4. This switches the signs on the f_3 and f_4 terms in the equation for n_z but leaves the other equations unchanged. Thus, the dynamics equations become the following:

$$
\begin{aligned}
f_W &= f_1 + f_2 + f_3 + f_4 \\
n_x &= -\ell f_1 + \ell f_3 \\
n_y &= \ell f_2 - \ell f_4 \\
n_z &= -k f_1 + k f_2 + k f_3 - k f_4
\end{aligned}
$$

Make a new matrix A in terms of ℓ and k based on this change. Would you be able to uniquely determine the individual forces on each propeller if you switch the spinning directions of propellers 3 and 4? Explain.

Solution: Linear dependence can be shown by Gaussian elimination, but we can also see by inspection that $R_4/k = (R_2 + R_3)/\ell$. Because of this linear dependence, some desired f_W and torques can no longer be achieved. This is why most quadcopters' propellers are set to spin as shown in 3.1.

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4. Batman (12 points)

You are the Batman. Your arch-nemesis, the Penguin, has released a swarm of 500 poisonous mechanical penguins in Gotham City. Each timestep, each penguin will move to a different location in Gotham City, cycling between Arkham Asylum, Wayne Industries, and the Batcave.

You have at your disposal 500 flying nanobots that are capable of turning off the penguins. However, each nanobot can only turn off one penguin, so we must send the exact number of nanobots as there are penguins at each location in order to defeat the Penguin.

Let a be the number of penguins in Arkham Asylum, w the number in Wayne Industries, and b the number in the Batcave.

We denote $\vec{x} = \begin{bmatrix} a \\ w \\ b \end{bmatrix}$.

(a) (8 points) Your casual acquaintance, Bruce Wayne, has provided you with two sensors to help you. Let t_1 denote the total number of penguins in Arkham Asylum and Wayne Industries, which is measured by sensor 1. Let t_2 denote the total number of penguins in Wayne Industries plus two times the number in the Batcave, which is measured by sensor 2.

i. (2 points) Write a system of equations to solve for the number of penguins at each location. Express it in matrix form, $\mathbf{A}\vec{x} = \vec{b}$.

Solution: Note: During the exam, it was clarified that for this part, we should not assume that we know the total number of penguins (i.e. 500 is not a valid number).

Writing out our system of equations:

$$a + w = t_1$$
$$w + 2b = t_2$$

Now, we can put this in matrix-vector form:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ w \\ b \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

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ii. (4 points) Find a basis for the nullspace of $\mathbf{A}$.
Solution: Solving for the nullspace, we set $\mathbf{A}\vec{x} = \vec{0}$. We select a as our free variable; let $a = s$.

$$a + w = 0 \rightarrow s + w = 0 \rightarrow w = -s$$

$$-s + 2b = 0 \rightarrow b = \frac{s}{2}$$

The basis for $\text{nul}(\mathbf{A})$ is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0.5 \end{bmatrix} \right\}$.

iii. (2 points) Given this nullspace, can we find the exact number of penguins at each location? Explain your answer.
Solution: Since $\mathbf{A}$ has a non-trivial nullspace, its columns are linearly dependent. Therefore, this system does not have a unique solution.

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(b) (4 points) Desperate, you turn to the Riddler for help. He promises to give you an antidote for the penguin poison if you prove the following:

Let $\mathbf{U}$ and $\mathbf{V}$ be $n \times n$ matrices. If $\mathbf{UV} = \mathbf{0}$, prove that every vector in $\text{col}(\mathbf{V})$ is in $\text{nul}(\mathbf{U})$.

Solution:

Let us express $\mathbf{V}$ and the $\mathbf{0}$ matrix in terms of their columns:

$$\mathbf{U} \cdot \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} \vec{0} & \cdots & \vec{0} \end{bmatrix}$$

We can break this into n equations, $\mathbf{U} \cdot \vec{v}_i = \vec{0}\ \forall i$. This means that each column of $\mathbf{V}$ is in the nullspace of $\mathbf{U}$. The column space of $\mathbf{V}$ is defined as any linear combination of the columns of $\mathbf{V}$. Since $\mathbf{U} \cdot \vec{v}_i = \vec{0}\ \forall i$, $\mathbf{U}$ times any linear combination of the columns of $\mathbf{V}$ will be 0. $\square$

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5. Intro to Intro to Quantum Computing (18 points)

Concepts from linear algebra show up everywhere, and one of the more interesting applications is in quantum mechanics. In particular, "spin", a fundamental property of a particle and the foundation of quantum computing, can be described using vectors. Your TAs Nick and Ryan want to make a quantum computer, and are trying to understand spin, but they need your help!

Spin states can be represented as a 2-element vector. A particle like an electron can have a state of either *spin up* (represented as $\vec{\chi}_+$), or *spin down* (represented as $\vec{\chi}_-$) which turn out to be eigenvectors of a special matrix called a spin matrix (represented as $\mathbf{S}$). Their corresponding eigenvalues are also important, and tell you the spin *value* for that particle. There are multiple spin matrices, but we will look at just two of them. We'll leave their significance for Physics 137A.

(a) (4 points) In order to help them out, we first want to explain to Nick and Ryan that to find the possible spin states and spin values for a given spin matrix, we just need to find the eigenvectors and eigenvalues of that matrix. Assume we are given the spin matrix $\mathbf{S_x}$, as shown below

$$\mathbf{S_x} = \begin{bmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{bmatrix}$$

where $\hbar$ is a constant.

Find the eigenvectors and eigenvalues of the above matrix.

Solution:

We start with the Eigenvalue Equation

$$\mathbf{S_x}\vec{\chi} = \lambda\vec{\chi}$$
$$(\mathbf{S_x} - \lambda\mathbf{I})\vec{\chi} = 0$$

Thus,

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = 0$$

This reduces to

$$\lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0$$

Thus we have

$$\lambda_+ = \frac{\hbar}{2}, \lambda_- = -\frac{\hbar}{2}$$

Plugging this back into the $\mathbf{S_x} - \lambda\mathbf{I}$ matrix and solving for the null space:

$$\left[\begin{array}{cc|c} -\frac{\hbar}{2} & \frac{\hbar}{2} & 0 \\ \frac{\hbar}{2} & -\frac{\hbar}{2} & 0 \end{array}\right]$$
$$\implies \vec{\chi}_+ = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{\hbar}{2} & \frac{\hbar}{2} & | & 0 \\ \frac{\hbar}{2} & \frac{\hbar}{2} & | & 0 \end{bmatrix}$$
$$\implies \vec{\chi}_{-} = \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

where α and β are constants. Note that you did not have to determine which vector was spin up vs. spin down or use the same variables. Just correctly computing both pairs is enough.

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(b) (5 points) Unfortunately, quantum mechanics has a lot of randomness. When you measure the spin, you will get one eigenvector or the other, but until then you can't know what you will get. This is because spin state vectors are always linear combinations of spin up/down vectors before measurement. To reiterate:

Any possible unmeasured spin state vector, $\vec{\gamma}$, is a linear combination of the spin up and spin down vectors ($\vec{\chi}_+$ and $\vec{\chi}_-$).

If we know $\vec{\gamma}$, then we can still determine which eigenvector we're more likely to get after we measure the spin. **If we have a state $\vec{\gamma} = a\vec{\chi}_+ + b\vec{\chi}_-$ then the likelihood of finding the particle in the spin up state is $|a|^2$ and is $|b|^2$ for the spin down state.**

Let's take a look at how this works with a different spin matrix $\mathbf{S_z}$, shown below.

$$\mathbf{S_z} = \begin{bmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{bmatrix}$$

with eigenvector/eigenvalue pairs

$$(\vec{\chi}_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda_+ = \frac{\hbar}{2}) \text{ and } (\vec{\chi}_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \lambda_- = -\frac{\hbar}{2})$$

Where $\hbar$ is a constant. Nick has an electron in an initial state $\vec{\gamma}$ that he hasn't measured yet and he wants to know what to expect.

For the initial unmeasured state $\vec{\gamma} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, determine which spin is more likely, up or down, and determine the probability ($|a|^2$ or $|b|^2$) for that spin.

Solution: We want to represent the initial state vector as a linear combination of the two eigenvectors of $\mathbf{S_z}$

To do this we can solve the vector equation

$$a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

By inspection, we can see that

$$a = \frac{3}{5}, b = \frac{4}{5}$$

So the likelihood of measuring spin up is $|a|^2 = \frac{9}{25}$ and the likelihood of measuring spin down is $|b|^2 = \frac{16}{25}$

Thus, we are more likely to find the electron in the spin down state.

It's the linear combination (or superposition) property of the electron as a bit that makes quantum computing so powerful. The qubit that the electron represents can be a 1 or a 0, but while it has not been formally measured, you can use the superposition itself as a means of storing information too! Namely, via the coefficients of the linear combination. Check out this link if you want to learn more about how qubits work! https://www.quora.com/What-exactly-is-a-qubit-and-how-does-it-work

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(c) (4 points) The previous part tells you what spin state (and therefore spin value) to expect after a single measurement, but sometimes we're more concerned with the average measurement.

For a real-valued $\vec{\gamma}$ you can determine what the average expected spin value is by computing $\vec{\gamma}^T \mathbf{S_z} \vec{\gamma}$.

Now Ryan wants to know what the average expected spin value of Nick's electron is.

What is the average expected spin value for the state $\vec{\gamma}$ in part (b)?

Solution: The information we can get from the average expected spin is the most information we can get out of the initial state without directly measuring the spin of the system.

We compute the average expected spin using the provided equation: $\vec{\gamma}^T \mathbf{S_z} \vec{\gamma}$

$$\vec{\gamma}^T \mathbf{S_z} \vec{\gamma} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} = -\frac{7}{25}\frac{\hbar}{2}$$

(d) (5 points) Now let's show something more general. In quantum mechanics, the energy of a system is given by Schrodinger's Equation,

$$\mathbf{H}\vec{\psi} = e\vec{\psi}$$

where e is a constant that represents the energy of the system, and is the eigenvalue corresponding to the state eigenvector $\vec{\psi}$. We often want to measure the energy of a system to determine which state it's in, but it's not always so simple.

Show that for matrix H, if two states $\vec{\psi}_1$ and $\vec{\psi}_2$ have the same eigenvalue e, then any linear combination of the two has the same eigenvalue e.

Solution: Given that $\mathbf{H}\vec{\psi}_1 = e\vec{\psi}_1$ and $\mathbf{H}\vec{\psi}_2 = e\vec{\psi}_2$

Then let $\vec{\psi} = \alpha_1\vec{\psi}_1 + \alpha_2\vec{\psi}_2$

$$\mathbf{H}\vec{\psi} = \mathbf{H}(\alpha_1\vec{\psi}_1 + \alpha_2\vec{\psi}_2) = \alpha_1\mathbf{H}\vec{\psi}_1 + \alpha_2\mathbf{H}\vec{\psi}_2 = \alpha_1 e\vec{\psi}_1 + \alpha_2 e\vec{\psi}_2 = e(\alpha_1\vec{\psi}_1 + \alpha_2\vec{\psi}_2) = e\vec{\psi}$$

An alternate solution is to note that since both have the same eigenvector, they must both be a part of the same eigenspace. Since subspaces are closed under scalar multiplication and vector addition, then any linear combination must still be in the same space, and therefore have the same eigenvalue.

Thus we have shown that any linear combination of the two states ψ_1 and ψ_2 is also an eigenvalue of $\mathbf{H}$.

The consequence of this is that it's not always possible to determine what's going on purely by measuring energy, which is usually possible in classical physics, and the state associated with eigenvectors for which this is true are called "degenerate states". This is one of the many quirks of quantum mechanics that physicists need to account for when measuring and doing research!

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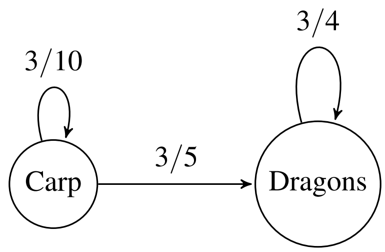
6. The Carp and the Dragons (16 points)

According to legend, there exists a magical river full of carp. Every year on January 1st, there is a special competition where each carp swims its hardest upstream, and those that successfully reach the top of the river become dragons. Every year on this day, $3/5$ of the carp succeed, $1/10$ perish during the arduous trip, and $3/10$ give up but escape with their lives intact. On the other hand, $1/4$ of the existing dragons die of old age. These transitions are represented in the diagram below:



(a) (2 point) If $\vec{x} = \begin{bmatrix} x_C \\ x_D \end{bmatrix}$ is a vector where x_C and x_D are the number of carp and dragons, express the change in $\vec{x}$ on January 1st as a matrix A. In other words, if $\vec{x}$ is the vector before the contest, then $\mathbf{A}\vec{x}$ is the vector after the contest.

Solution:

$$\mathbf{A} = \begin{bmatrix} 3/10 & 0 \\ 3/5 & 3/4 \end{bmatrix}$$

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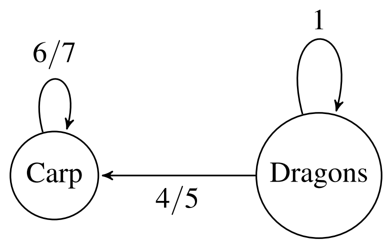
(b) (2 point) In addition to the contest, over the course of one year, the dragons reproduce, and $4/5$ of the dragons give birth to one carp each (dragons can only give birth to carp). On the other hand, $1/7$ of the carp perish due to the harsh river conditions. The transitions are represented in the diagram below:



Write down a matrix **B** representing this change.

Solution:

$$\mathbf{B} = \begin{bmatrix} 6/7 & 4/5 \\ 0 & 1 \end{bmatrix}$$

(c) (4 points) Regardless of your previous answers, let $\mathbf{A} = \begin{bmatrix} 1/5 & 0 \\ 7/10 & 3/4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1/3 & 1/3 \\ 0 & 1 \end{bmatrix}$. Suppose that every year, we have the contest in (a) on Jan 1st, as well as the reproduction in (b) that occurs during the rest of the year. If $\vec{x}$ is the carp-dragon vector on December 31st, 2019, find the matrix **C** such that $\mathbf{C}\vec{x}$ is the carp-dragon vector on December 31st, 2020.

Solution: The transition matrix for one year is

$$\mathbf{C} = \mathbf{BA} = \begin{bmatrix} 1/3 & 1/3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/5 & 0 \\ 7/10 & 3/4 \end{bmatrix} = \begin{bmatrix} 3/10 & 1/4 \\ 7/10 & 3/4 \end{bmatrix}$$

Note that the contest happens first, which determines the multiplication order.

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(d) (8 points) Regardless of your answers above, let $\mathbf{C} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$. Suppose the pattern in (c) repeats every year, and we start with 900 carp and 0 dragons on December 31, 2019. Will the number of carp and dragons settle to some steady state? Please justify your answer. Furthermore, if the numbers do stabilize, please find the number of carp and dragons on December 31st in some year far into the future.

Solution: To determine the existence of a steady state, we can find the eigenvalues by the characteristic polynomial, or

$$(\lambda - 1/2)(\lambda - 3/4) - 1/8 = \lambda^2 - (5/4)\lambda + 1/4 = (\lambda - 1)(\lambda - 1/4)$$

Note that one of the eigenvalues is 1 while the other is 1/4, which has magnitude less than 1. Therefore, there will be a steady state.

To find the steady state, we can find the nullspace of $\mathbf{C} - \mathbf{I}$:

$$\begin{bmatrix} -1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}.$$

The eigenvector is $[1/2, 1]$, which when normalized becomes $[1/3, 2/3]$. Therefore, we have 300 carp and 600 dragons.

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7. Three Dimensions of a Virtual Reality System (19 points)

A virtual reality (VR) game developer is attempting to build a set of basis vectors to represent every point in $\mathbb{R}^3$.

(a) (6 points) After coming up with a set of three basis vectors in $\mathbb{R}^3$ and storing them in a 3×3 matrix $\mathbf{A}$, a power outage erased one element of every basis vector, i.e.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & t \\ t & 1 & -1 \\ 1 & t & -1 \end{bmatrix},$$

where $t \in \mathbb{R}$ represents the number that was erased. Due to program restrictions, only one unique value for t can be substituted back into the erased elements to recover the set of basis vectors. The game developer is however not familiar with linear algebra. Using your understanding of linear independence, **provide the game developer with the set of all possible real number values of t that they can use to ensure that A's columns make up a set of basis vectors.**

Solution: We can perform Gaussian Elimination on $\mathbf{A}$ to recover the set of basis vectors.

$$\begin{bmatrix} 1 & 0 & t \\ t & 1 & -1 \\ 1 & t & -1 \end{bmatrix} \overbrace{\longrightarrow}^{R_2 \leftarrow R_2 - tR_1} \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & -(t^2+1) \\ 1 & t & -1 \end{bmatrix} \overbrace{\longrightarrow}^{R_3 \leftarrow R_3 - R_1} \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & -(t^2+1) \\ 0 & t & -(t+1) \end{bmatrix} \overbrace{\longrightarrow}^{R_3 \leftarrow R_3 - tR_2} \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & -(t^2+1) \\ 0 & 0 & t^3-1 \end{bmatrix}.$$

In order for $\mathbf{A}$ to have linearly independent columns, the last row of the row-reduced matrix from Gaussian Elimination cannot be a row of all zeros. In other words, $t^3 \neq 1 \;\Rightarrow\; t \neq 1$. Therefore, as long as $t \neq 1$, we have the required set of linearly independent basis vectors.

Alternative Solution:

Another valid solution was to note that $\det(A) \neq 0$. This produces

$$\det(A) = 1 * \begin{vmatrix} 1 & -1 \\ t & -1 \end{vmatrix} - 0 * \begin{vmatrix} t & -1 \\ 1 & -1 \end{vmatrix} + t * \begin{vmatrix} t & 1 \\ 1 & t \end{vmatrix} = (-1-(-1)t) + t(t^2-1) = -1 + t + t^3 - t = t^3 - 1 \neq 0$$

$$\rightarrow t^3 \neq 1 \rightarrow t \neq 1$$

Common Mistakes:

Full credit was only given to both the correct answer and the correct reasoning. Many people noted that $t \neq 1$ through observation and noting that this made A linearly dependent. However, this does not prove that t can be any value other than 1, just that t cannot be 1.

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(b) (7 points) The game developer now decides to test the game on the VR headset, which operates using a proprietary processor chip. In order to operate at a faster speed using lower power consumption, the processor calculates its own set of basis vectors $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ using the developer's basis vectors as inputs. It denotes the input basis vectors as some arbitrary vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, and then calculates its own set of three basis vectors $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ such that

$$\begin{aligned}\vec{w}_1 &= \vec{v}_1\\ \vec{w}_2 &= b\vec{v}_3 - \vec{v}_2\\ \vec{w}_3 &= \vec{v}_1 + a\vec{v}_2 + \vec{v}_3,\end{aligned}$$

where $a, b \in \mathbb{R}$. The VR headset operating system asks that you specify what a and b must be so that $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ make up a set of basis vectors given any arbitrary basis vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. Knowing that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are three arbitrary vectors that are linearly independent, **provide the game developer with the set of all real number values of a and b such that $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ make up a set of basis vectors.** [Hint: the scalars a and b multiply every element of the vectors $\vec{v}_2$ and $\vec{v}_3$, respectively, so it is possible to represent every vector as a scalar symbolically, and compactly write the system as follows:]

$$\begin{bmatrix}(\vec{w}_1)\\(\vec{w}_2)\\(\vec{w}_3)\end{bmatrix} = \mathbf{A}\begin{bmatrix}(\vec{v}_1)\\(\vec{v}_2)\\(\vec{v}_3)\end{bmatrix},$$

where $\mathbf{A}$ is a 3×3 matrix.

Solution: Because the scalars a and b multiply every element of the vectors $\vec{v}_2$ and $\vec{v}_3$, respectively, we can represent the above system of linear equations as follows:

$$\begin{bmatrix}(\vec{w}_1)\\(\vec{w}_2)\\(\vec{w}_3)\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & -1 & b\\1 & a & 1\end{bmatrix}\begin{bmatrix}(\vec{v}_1)\\(\vec{v}_2)\\(\vec{v}_3)\end{bmatrix}.$$

Denote by $\mathbf{A}$ the 3×3 matrix above, i.e.

$$\mathbf{A} = \begin{bmatrix}1 & 0 & 0\\0 & -1 & b\\1 & a & 1\end{bmatrix}.$$

We have that $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ are linearly independent if $\mathbf{A}$'s columns are linearly independent since $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are linearly independent. We perform Gaussian Elimination on $\mathbf{A}$ to determine the set of all values for a and b that make $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ linearly independent. Performing Gaussian Elimination, we have

$$\begin{bmatrix}1 & 0 & 0\\0 & -1 & b\\1 & a & 1\end{bmatrix} \overset{R_3 \leftarrow R_3 - R_1}{\overbrace{\longrightarrow}} \begin{bmatrix}1 & 0 & 0\\0 & -1 & b\\0 & a & 1\end{bmatrix} \overset{R_3 \leftarrow R_3 + aR_2}{\overbrace{\longrightarrow}} \begin{bmatrix}1 & 0 & 0\\0 & -1 & b\\0 & 0 & 1+ab\end{bmatrix}.$$

Similar to Part (a), in order for $\mathbf{A}$ to have linearly independent columns, the last row of the matrix we obtained from Gaussian Elimination cannot be a row of all zeros. In other words, $ab \neq -1$. Therefore, as long as $ab \neq -1$, we have the required set of linearly independent basis vectors $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$.

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(c) (6 points) For this part, let $a = 1$ and $b = -2$ in $\mathbf{A}$ from Part (b) [Hint: the pair of values $a = 1$ and $b = -2$ should be included in your answer to Part (b), i.e.]

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}.$$

In other words, assume that the values for a and b comprise the following system of equations:

$$\begin{bmatrix} (\vec{w}_1) \\ (\vec{w}_2) \\ (\vec{w}_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} (\vec{v}_1) \\ (\vec{v}_2) \\ (\vec{v}_3) \end{bmatrix}.$$

The game developer would now like to have a matrix $\mathbf{B}$ that they can use to recover $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ given $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ via a simple matrix vector multiply. To elaborate, the game developers would like a 3×3 matrix $\mathbf{B}$ that satisfies the following:

$$\begin{bmatrix} (\vec{v}_1) \\ (\vec{v}_2) \\ (\vec{v}_3) \end{bmatrix} = \mathbf{B} \begin{bmatrix} (\vec{w}_1) \\ (\vec{w}_2) \\ (\vec{w}_3) \end{bmatrix}.$$

What is B?

Solution: Observe that the matrix we need is $\mathbf{A}^{-1}$. After substituting in the given values for a and b in $\mathbf{A}$ from Part (b), we can obtain $\mathbf{B} = \mathbf{A}^{-1}$ using Gaussian Elimination after augmenting the identity matrix as follows:

$$\left[\ \mathbf{A}\ \middle|\ \mathbf{I}\ \right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R_3 \leftarrow R_3 - R_1 + R_2}{\widehat{\longrightarrow}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow -R_3}{\widehat{\longrightarrow}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right] \overset{R_2 \leftarrow R_2 + 2R_3}{\widehat{\longrightarrow}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right]$$

$$\overset{R_2 \leftarrow -R_2}{\widehat{\longrightarrow}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right] = \left[\ \mathbf{I}\ \middle|\ \mathbf{B}\ \right].$$

Therefore,

$$\mathbf{B} = \mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}.$$

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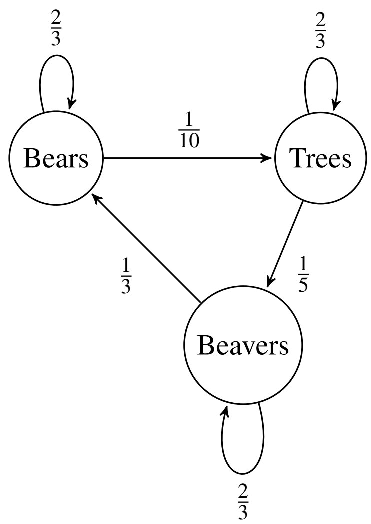
8. Population Stabilization(21 points)

After a large forest fire, an area of the California forest has burnt down. We would like to reintroduce populations of Bears, Trees, and Beavers into the burnt down forest. However, we need to be careful how many of each species to introduce into the area. Luckily, a team of conservationists have observed the number of each animal over time, and come up with a model shown below:



(a) (3 Points) Given the state vector $\vec{x}$ shown below, we can represent the evolution of the system with the matrix equation shown below:

$$\vec{x}[k] = \begin{bmatrix} x_r[k] \\ x_t[k] \\ x_v[k] \end{bmatrix}, \vec{x}[k+1] = \mathbf{A}\vec{x}[k]$$

where $x_r[k]$, $x_t[k]$ and $x_v[k]$ represents the number of Bears, Trees, and Beavers at a specific time step t. Given the diagram above, find the matrix $\mathbf{A}$ that represents this system. Is this system conservative?

Solution: The matrix $\mathbf{A}$ is not conservative, as not all the columns sum to 1.

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(b) (10 Points) It turns out, an intern made a mistake counting the number of Bears. The conservationists went back and updated their model, this time they have provided you with a matrix **B** that represents the system.

$$\vec{x}[k+1] = \begin{bmatrix} x_r[k+1] \\ x_t[k+1] \\ x_v[k+1] \end{bmatrix} = \mathbf{B} \begin{bmatrix} x_r[k] \\ x_t[k] \\ x_v[k] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_r[k] \\ x_t[k] \\ x_v[k] \end{bmatrix}$$

Find the eigenvalues and eigenvectors of this matrix.

Solution: We begin by finding the eigenvalues of the above matrix.

$$det(\mathbf{A} - \lambda I) = det(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & \frac{1}{2}-\lambda & 1 \\ 0 & 1 & \frac{1}{2}-\lambda \end{bmatrix}) = (1-\lambda)((\frac{1}{2}-\lambda)^2 - 1) = (1-\lambda)(\frac{1}{2}+\lambda)(\frac{3}{2}-\lambda)$$

Next we find the eigenvectors associated with each eigenvalue: $\lambda_1 = 1$:

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$$

From the above matrix, we see x_r is free and x_t and x_v must be zero, therefore $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Next for the eigenvalue $\lambda_2 = -\frac{1}{2}$:

$$A + \frac{1}{2}I = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_v$is free

$$x_v = x_v$$

$$x_t = -x_v$$

$$x_r = 0$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Finally for the eigenvalue $\lambda_3 = \frac{3}{2}$:

$$A - \frac{3}{2}I = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_v$is free

$$x_v = x_v$$

$$x_t = x_v$$

$$x_r = 0$$

$$v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

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(c) (8 Points) It turns out the conservationists made a mistake again. Apologetic, this time they provide you with the eigenvalues and eigenvectors of the matrix $\mathbf{C}$ that represents the system:

$$\left(\lambda_1 = 2, \vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}\right), \left(\lambda_2 = 1, \vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}\right), \left(\lambda_3 = \frac{1}{2}, \vec{v}_3 = \begin{bmatrix}1\\1\\2\end{bmatrix}\right)$$

You are trying to select the number of the bears, trees, and beavers to introduce in the forest, that is, you are trying to find a vector $\vec{x}[0]$.

i. (4 Points) Describe the set of vectors V_1 such that if $\vec{x}[0]$ was an element of V_1, then the total number of animals in this system ~~to~~ would eventually go to 0. Does the set of vectors form a subspace of $\mathbb{R}^3$? Simply state whether the set of vectors form a subspace or not, you do not need to provide a proof.

Solution: In order for the system to decay to 0, we need the initial state to be in the span of eigenvectors corresponding to eigenvalues less than 1, therefore the set V_1 is span $\left(\begin{bmatrix}1\\1\\2\end{bmatrix}\right)$. Since this set is the span of a vector, it is a subspace of $\mathbb{R}^3$.

ii. (4 Points) Describe the set of vectors V_2 such that if $\vec{x}[0]$ was an element of V_2, then the total number of animals in this system grows unbounded. Does the set of vectors form a subspace of $\mathbb{R}^3$? Simply state whether the set of vectors form a subspace or not, you do not need to provide a proof.

Solution: In order for the system to grow unbounded, we need to the initial state to include some component in the direction of the eigenvector corresponding to eigenvalue greater than 1. Therefore the set V_2 can be described as: $\left\{\vec{x} \middle| \vec{x} = \alpha \begin{bmatrix}1\\1\\1\end{bmatrix} + \beta \begin{bmatrix}0\\1\\1\end{bmatrix} + \gamma \begin{bmatrix}1\\1\\2\end{bmatrix}, \alpha \neq 0\right\}$. Notice this set of vectors does not include the zero vector, and therefore is not a subspace of $\mathbb{R}^3$