Problem # 1 ($1 \times 8 = 8$ points)

For each statement, state whether the claim is True or False.
Circle you answer. No explanation is necessary.
-1 **points for incorrect answers** so guessing is not advised.

1 **True** or **False** First-order linear systems can have an oscillatory free-response.
False. You need complex poles for an oscillatory free response. So the system must be second order or higher.

2 **True** or **False** If two square matrices A and B have the same eigenvalues, then $A = B$.
False. Counterexample:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

3 **True** or **False** Proportional control can always stabilize a first-order LTI plant.
True. Easy to check.

4 **True** or **False** Anti-wind up strategies are needed when using **pure** proportional control.
False. Anti-wind up strategies are needed for integral control.

5 **True** or **False** If you have a bad plant model, feedback linearization is a bad idea.
Not graded, as it is ut of the scope of Midterm 2.

6 **True** or **False** Increasing the proportional gain k_p, reduces the time constant.
True. Increasingk_p moves the pole deeper into the left half plane which gives a smaller time constant.

7 **True** or **False** Suppose the realization $\Sigma(A, B, C, D)$ is stable.. Then A is invertible.

8 **True** or **False** The steady-state response of a stable linear system due to a sinusoidal input depends on the initial conditions.
False. Steady-state response never depends on initial conditions.

Problem # 2 (2 +2+2+3+3+3 = 15 points)

(a) Give any two reasons why state-space methods are very powerful.
(a) High order plants, multi-input multi-output plants, (c) control design computations are easy and systematic.

(b) What is the single most important reason to use integral control?
Reject constant disturbances.

(c) When should we use anti-windup control strategies?
When the plant has actuator saturation and you are using integral control.

(d) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} p & 1 & 0 \\ 0 & q & 0 \\ 1 & 1 & r \end{bmatrix}$$

Eigenvalues are p, q, r.

(e) Calculate $\sin(A)$ for the matrix

$$A = \begin{bmatrix} \pi & 100 \\ 0 & 2\pi \end{bmatrix}$$

Notice that the eigenvalues of A are $\pi, 2\pi$ because it is upper triangular. Then

$$\sin(A) = T \begin{bmatrix} \sin(\pi) & 0 \\ 0 & \sin(2\pi) \end{bmatrix} T^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

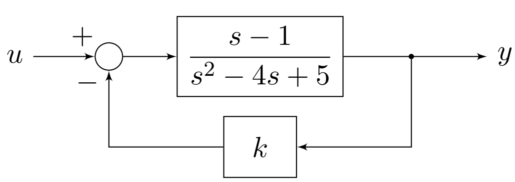
(f) Find the DC gain matrix of the transfer function

$$P(s) \sim \left[\begin{array}{cc|ccc} -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\text{DC gain } = -CA^{-1}B + D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Problem # 3 (4 + 4 +4 = 12 points)

(a) Consider the feedback system shown below.



For what values of k is this feedback system stable?

Closed loop transfer function from u to y is

$$y = \left[\frac{s-1}{s^2 + (k-4)s + (5-k)}\right] u$$

For stability we need $4 < k < 5$.

(b) Consider the plant with transfer function

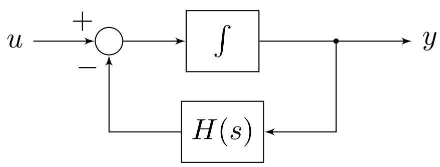
$$H(s) = \frac{s^2 + cs + d}{s^2 + 4s + 4}$$

We apply the input $u(t) = \sin(2t)$. The steady-state response is zero. Find c and d.

Since the steady state response is zero, we must have $H(j2) = 0$. So we get

$$0 = H(j2) = \frac{-4 + 2cj + d}{*} \quad \Longrightarrow \quad d = 4, \quad c = 0$$

(c) Consider the feedback system shown below



$$H(s) \sim \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right]$$

Find the transfer function from u to y.

Notice that $H(s) = 1/s$. So the closed loop transfer function is

$$T(s) = \frac{1/s}{1 + 1/s^2} = \frac{s}{s^2 + 1}$$

Problem # 4 (12 points)

Below are 6 different realizations.

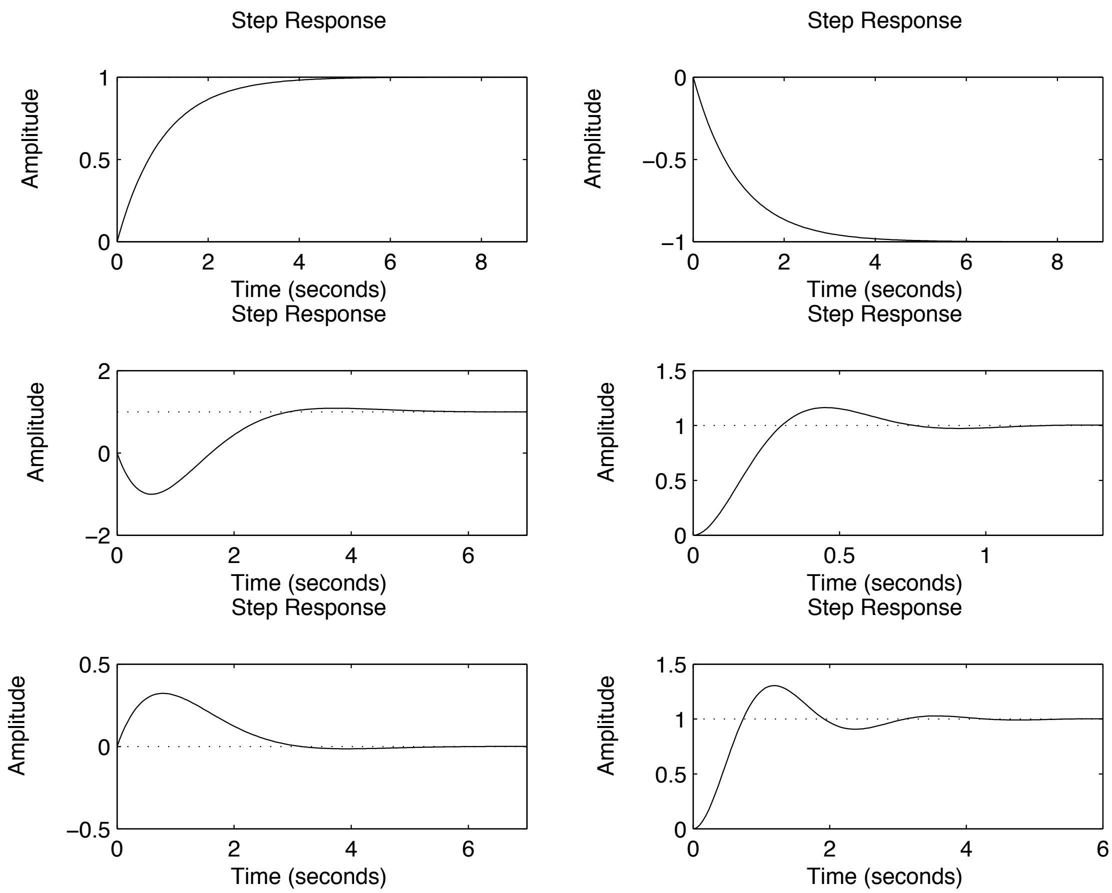
A. $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -8 & -2 & 1 \\ \hline 8 & 0 & 0 \end{array}\right]$ D. $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -64 & -8 & 1 \\ \hline 64 & 0 & 0 \end{array}\right]$

B. $\left[\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right]$ E. $\left[\begin{array}{c|c} -1 & 1 \\ \hline -1 & 0 \end{array}\right]$

C. $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -2 & 1 \\ \hline 2 & -4 & 0 \end{array}\right]$ F. $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -2 & 1 \\ \hline 0 & 1 & 0 \end{array}\right]$

The unit-step response, starting from zero initial conditions at $t = 0$ for these models are shown on the next page. Match these step responses with the models by entering the letters **A** through **F** in the boxes provided.

All the given realizations are in controllable canonical form. So we can read off the transfer functions. We get:



A. $\dfrac{8}{s^2+2s+8}$

B. $\dfrac{1}{s+1}$

C. $\dfrac{-4s+2}{s^2+2s+2}$

D. $\dfrac{64}{s^2+8s+64}$

E. $\dfrac{-1}{s+1}$

F. $\dfrac{s}{s^2+2s+2}$

All these transfer functions are stable.
Only E. has a DC gain of -1, so it must be the step response 4.
Only C. has a right-half-plane zero, so its step response goes the wrong way first, so C. = 2.
Only F. has a DC gain of 0, so it must be the step response 3.
The remaining step responses are 1, 5, 6.
Of these, 5. and 6. have overshoot, so they cannot be first-order systems. So B. = 1.
The settling time of the step response 5. is much less than that of 6.
Thus, D. = 5. By elimination, A = 6.
In summary, $(A, B, C, D, E, F) = (6, 1, 2, 5, 4, 3)$.

Problem # 5 (5+5 = 10 points)

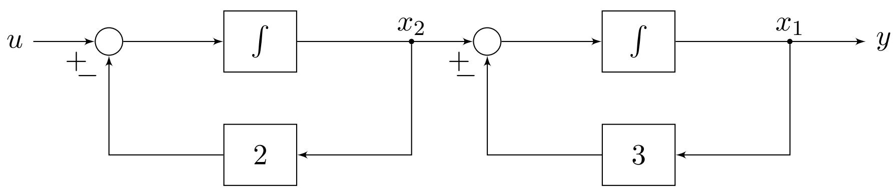
(a) Consider the feedback system shown below.
Find a realization for the transfer function $H(s)$ from u to y, i.e. find matrices A, B, C, D such that

$$H(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Use the state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where x_1 and x_2 are the signals shown.



Using the state variables x_1, x_2 as shown, we get

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

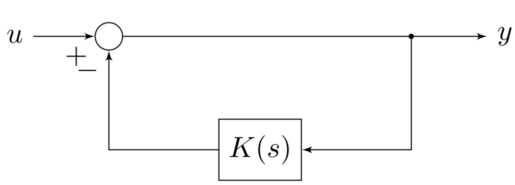
where x_1 and x_2 are the signals shown.

(b) Consider the feedback system shown below.
Suppose $K(s)$ has state-space realization

$$K(s) \sim \left[\begin{array}{c|c} F & G \\ \hline H & 0 \end{array}\right]$$

Compute any state space realization for the transfer function $H(s)$ from u to y, i.e. find matrices A, B, C, D such that

$$H(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$



Let e be the output of $K(s)$. From the block diagram, we can write

$$\dot{x} = Fx + Gy, \quad y = u - e = u - Hx$$

These can be combined as

$$\dot{x} = (F - GH)x + Gu, \quad y = -Hx + Iu \implies \text{closed-loop system} \sim \left[\begin{array}{c|c} F - GH & G \\ \hline -H & I \end{array}\right]$$

Problem # 6 (2 + 2 + 2 +2 = 8 points)

Consider the LTI system with input u and output y modeled by

$$y(t) = u(t-2)$$

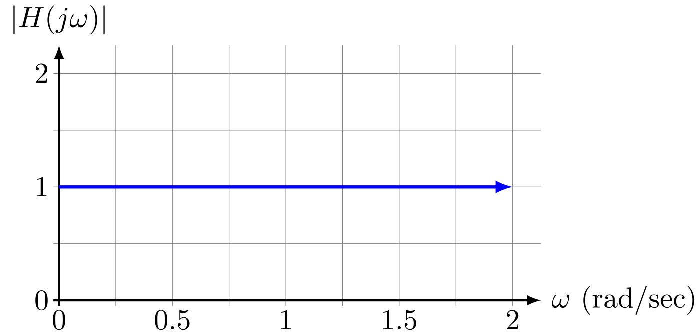
(a) Find the transfer function $H(s)$ from u to y.

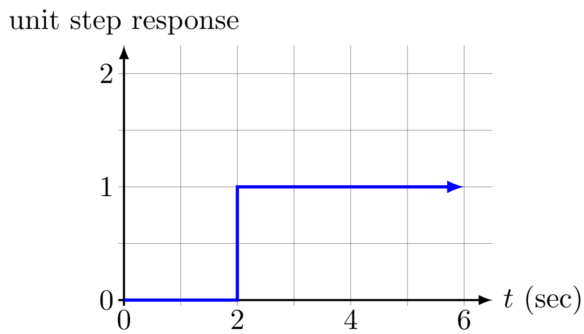
(b) Plot the unit step response.

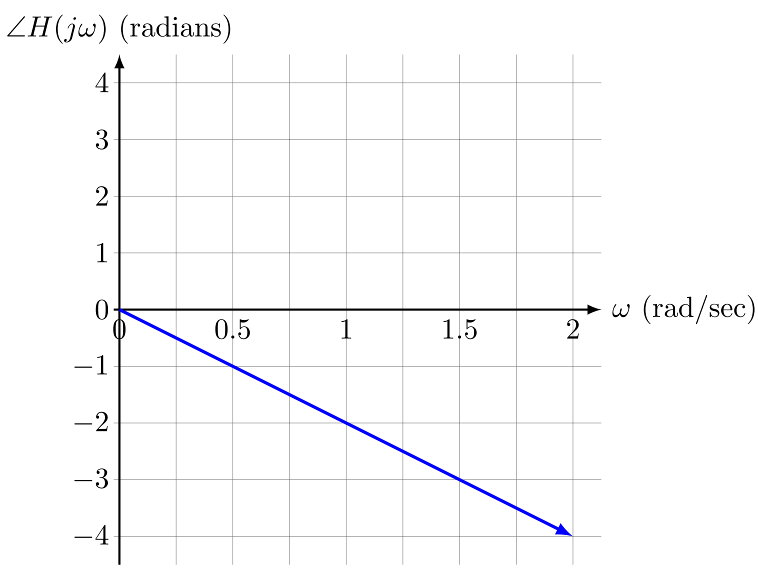
(c) Plot the magnitude frequency response of $H(s)$.

(d) Plot the phase frequency response of $H(s)$.

To make it easier, the scales are linear (not log-frequency or decibels).







(a) $H(s) = e^{-2s}$.

(b) The unit step response of $H(s)$ is a unit step function delayed by 2 seconds.

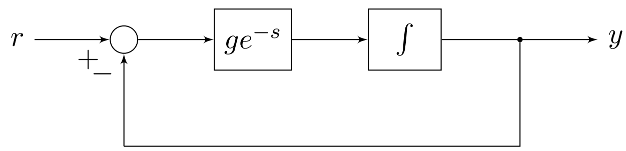
(c) $|H(j\omega)| = |e^{-2j\omega}| = 1$ for all ω.

(d) $\angle H(j\omega) = \angle e^{-2j\omega} = -2\omega$.

Problem # 7 (12 points)

For what values of g is the feedback system shown below stable?
Your answer should be in the form

$$g_{\min} < g < g_{\max}$$



You are asked to find the gain margin analytically. The nominal loop-gain is

$$L^o(s) = \frac{e^{-s}}{s}$$

The frequency response magnitude and phase of this transfer function are

$$|L^o(j\omega)| = \left|\frac{e^{-j\omega}}{j\omega}\right| = \frac{1}{\omega}, \quad \angle L^o(j\omega) = -\omega - \frac{\pi}{2} = \frac{3\pi}{2} - \omega$$

The phase crossover frequency (there is only one) is when the phase is π, or

$$\omega_c = \frac{\pi}{2}$$

At this cross over frequency, the loop gain magnitude is

$$|L^o(j\omega)| = \frac{1}{\omega_c} = \frac{2}{\pi}$$

So the gain margin is $[2/\pi, \infty)$ which means the feedback system is stable for

$$\frac{2}{\pi} < g < \infty$$