EECS 16B Designing Information Devices and Systems II
Fall 2019 UC Berkeley

Midterm 1

Exam location: NA

PRINT your student ID: ____

PRINT AND SIGN your name: ________________, ________________ ____________________
(last) (first) (signature)

PRINT your discussion section and GSI (the one you attend): ____

Row Number (front row is 1): __________ Seat Number (left most is 1): __________

Name and SID of the person to your left: ____

Name and SID of the person to your right: ____

Name and SID of the person in front of you: ____

Name and SID of the person behind you: ____

Section 0: Pre-exam questions (3 points)

1. **What classes are you taking this term? (1 pt)**

2. **Tell us about something that makes you happy (2 pts)**

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

3. High Speed Digital Communication (16 pts)

In this problem, we will analyze a simplified model of a USB communication link and show that there is a limit to how quickly you can transfer data.

As Figure 1 illustrates, the transmitter is a CMOS inverter, whose input is driven with the voltage source V_{in} representing the data to be sent over the link. The USB cable connects the transmitter to the receiver. The link successfully transfers data when the voltage at the output of the USB cable crosses the threshold of the receiver's inverter, thereby flipping the receiver output voltage $V_{out,Receiver}$.

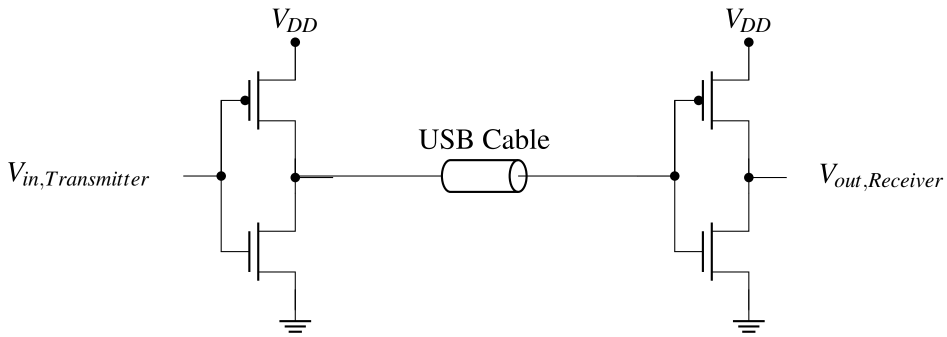


Figure 1: A USB communication link

To simplify, we will model the transmitter's inverter as a resistance R_{inv} and use $\bar{V}_{in}(t)$ as the digitally flipped voltage that represents the ideal output of the transmitter's inverter. The receiver inverter is modeled as an input capacitance C_{inv}. The cable is modeled as an RC system whose R_{wire} and C_{wire} values grow as the cable length increases. A diagram of the circuit model to be used in this problem is shown in Figure 3.

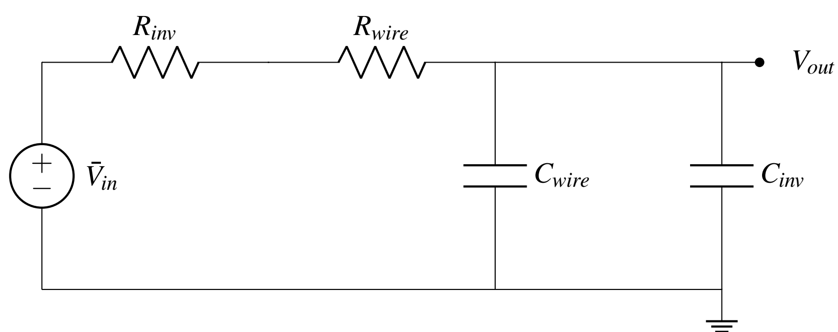


Figure 2: Simplified circuit model for a USB communication link

(a) (2 pts) **If** $\bar{V}_{in} = 0V$ **for all** $t < 0$ **what is** V_{out} **at time** $t = 0$**?** This will serve as the initial condition for the rest of the problem.

(b) (5 pts) **Write the differential equation for the voltage $V_{out}(t)$ as a function of $\bar{V}_{in}(t)$ for $t \geq 0$ in terms of $R_{inv}, C_{inv}, R_{wire}$, and C_{wire}.**

(c) (5 pts) For the rest of this problem, for ease of computation (since you don't have calculators), assume that $R_{inv} = 1\text{k}\Omega$, $R_{wire} = 3\text{k}\Omega$, $C_{wire} = 24\text{pF}$, and $C_{inv} = 1\text{pF}$. (Here, $\text{pF} = 10^{-12}\text{F}$ and $\text{k}\Omega = 10^3\Omega$.)

Now assume that $\bar{V}_{in}$ for time $t \geq 0$ is a piecewise-constant voltage source. $\bar{V}_{in}$ rises to 1V at $t = 0$ and then falls back down to 0V at some time $t = T_{bit}$. **Write an exact expression for $V_{out}(t)$ during the time period $0 \leq t < T_{bit}$.**

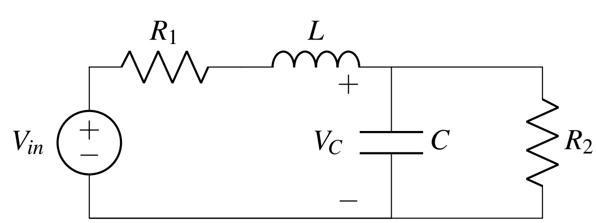
Your expression should be an explicit formula. No integrals are allowed to remain for full credit.

(HINT: Recall that $x_0 e^{\lambda t} + \int_0^t e^{\lambda(t-\tau)} u(\tau) d\tau$ is the unique solution of $\frac{d}{dt}x(t) = \lambda x(t) + u(t)$ with initial condition $x(0) = x_0$. You also might want to draw $\bar{V}_{in}(t)$ to help yourself.)

(d) (4 pts) **Sketch $V_{out}(t)$ during the time period $0 \leq t < T_{bit}$, clearly marking the initial voltage and the final voltage.**

4. RLC Circuits (30 pts)

Consider the following circuit:



If we define the state vector $\vec{x}(t) = \begin{bmatrix} V_C \\ I_L \end{bmatrix}$, standard circuit analysis would reveal that this circuit is governed by this system of differential equations:

$$\frac{d}{dt}\begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{R_2 C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_1}{L} \end{bmatrix}}_{A} \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}_{\vec{b}} V_{in}(t).$$

For this problem generally, it may help to recall the standard formula for a 2×2 matrix inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(a) (12 pts) Suppose that we choose to drive the circuit with a $V_{in}(t)$ that is a sinusoidal waveform at the angular frequency ω radians/sec. Let us consider the voltage $V_C(t)$ across the capacitor C as our output voltage $V_{out}(t)$. **What is the transfer function of this circuit, namely** $H(j\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}}$ where $\widetilde{V}_{in}$ is the input voltage phasor at angular frequency ω and $\widetilde{V}_{out}$ is the output voltage phasor at that same angular frequency ω.

Your answer should be symbolic in terms of the R_1, L, C, and R_2 along with $j = \sqrt{-1}$ and ω. You don't have to simplify this to look nice.

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

(b) (8 pts) Assume that initially the state is at rest, with the capacitor charged to $1V$, so $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Consider the values $C = 1, L = 1, R_1 = 1, R_2 = 1$. For convenience, we have plugged in the values as well as computed eigenvalues and eigenvectors for you. This yields an A matrix with eigenvalues $\lambda_1 = -1 + j$ and $\lambda_2 = -1 - j$.

$$\frac{d}{dt}\begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{b}} V_{in}(t), \quad \text{and eigenvectors } \vec{v}_{\lambda=-1+j} = \begin{bmatrix} -j \\ 1 \end{bmatrix}, \vec{v}_{\lambda=-1-j} = \begin{bmatrix} j \\ 1 \end{bmatrix}$$

For convenience, please note that if $V = [\vec{v}_{\lambda=-1+j}, \vec{v}_{\lambda=-1-j}]$, then $V^{-1} = \begin{bmatrix} -j & j \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{j}{2} & \frac{1}{2} \\ -\frac{j}{2} & \frac{1}{2} \end{bmatrix}$.

Change coordinates of this differential equation to be in terms of $\vec{\tilde{x}}(t) = V^{-1}\vec{x}(t)$ and the input $V_{in}(t)$. i.e. give an equation in the form $\frac{d}{dt}\vec{\tilde{x}}(t) = \widetilde{A}\vec{\tilde{x}}(t) + \vec{\tilde{b}} V_{in}(t)$.

What is the matrix $\widetilde{A}$, the vector $\vec{\tilde{b}}$ and the intial condition $\vec{\tilde{x}}(0)$?

(c) (10 pts) For the problem as stated in the previous part, **solve for the transient solution $\vec{x}(t)$ given the specified initial condition under the assumption that $V_{in}(t) = 0$ for all $t \geq 0$.**

Your solution should be expressed in terms of real functions of time t. No j's are permitted in your final answer for full credit.

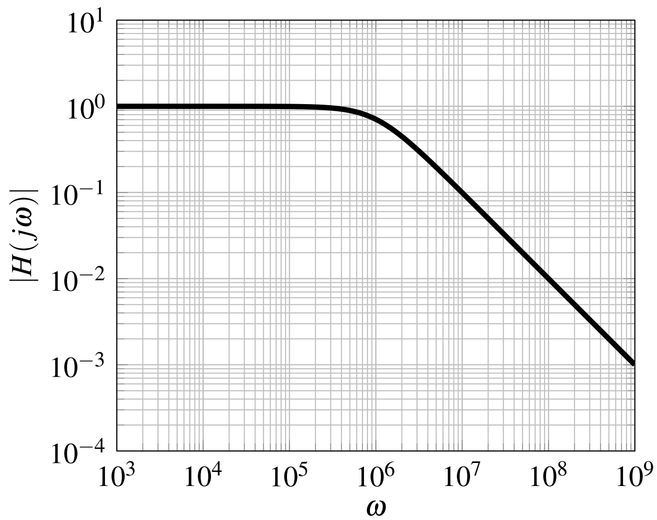
5. Transfer Functions and Filters (18 pts)

(a) (6 pts) **Identify each of the Bode Plots, circuits, and transfer functions as either a lowpass or highpass filter.** Indicate your answer by filling in the appropriate bubble.

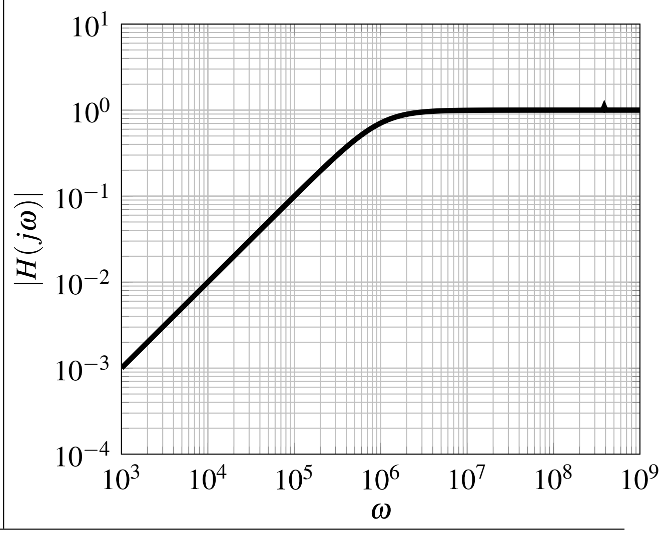
Table 1: Table to be filled in for your answers. Fill in bubbles.

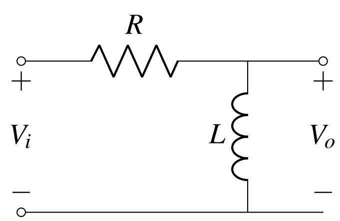
	Lowpass	Highpass
Bode Plot A	◯	◯
Bode Plot B	◯	◯
Circuit C	◯	◯
Circuit D	◯	◯
Transfer Fn E	◯	◯
Transfer Fn F	◯	◯

Bode Plot A ($\omega_c = 10^6$)

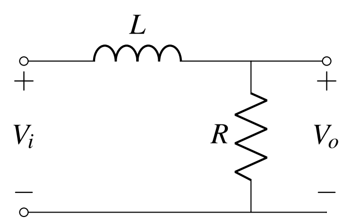


Bode Plot B ($\omega_c = 10^6$)





Circuit C: $H(j\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$



Circuit D: $H(j\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$

Transfer function E: $H_E(j\omega) = \dfrac{\frac{j\omega}{\omega_c}}{1 + \frac{j\omega}{\omega_c}}$ | Transfer function F: $H_F(j\omega) = \dfrac{1}{1 + \frac{j\omega}{\omega_c}}$

(b) (6 pts) Consider the three filters in cascade below, with unity-gain op-amp buffers in between them:

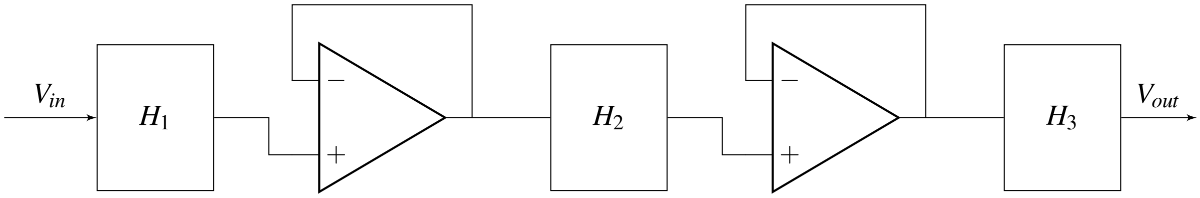


Figure 5: Three filters cascaded via unity-gain op-amp buffers

Suppose that at some frequency ω_0 radians/sec we know that:

$$H_1(j\omega_0) = 3e^{j\frac{\pi}{4}} \quad H_2(j\omega_0) = \frac{1}{2}e^{-j\frac{\pi}{3}} \quad H_3(j\omega_0) = 4e^{j\frac{5\pi}{6}}$$

If $V_{in}(t) = 2\sin\left(\omega_0 t + \frac{\pi}{2}\right)$:

What is the phasor for the input voltage: $\widetilde{V_{in}}$?

What is the phasor for the output voltage: $\widetilde{V_{out}}$?

What is $V_{out}(t)$?

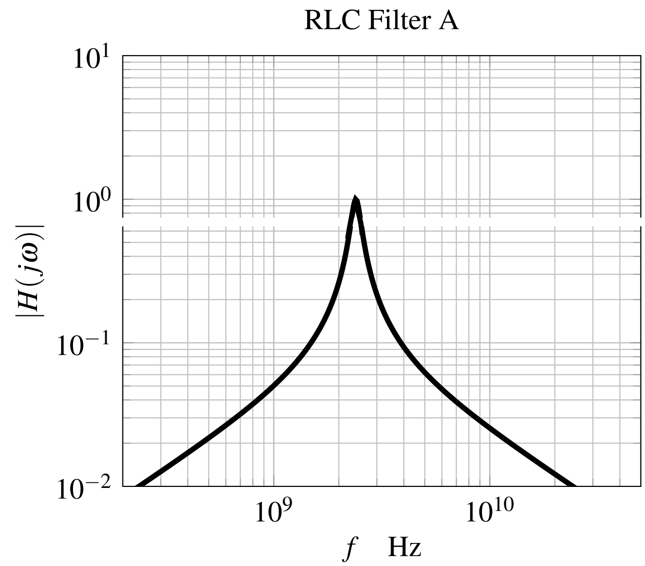
(c) (6 pts) Suppose there is an interfering signal at 5GHz that you need to get rid of, while passing through your WiFi signal at 2.4GHz. You have access to the following six components (two capacitors, two inductors, and two resistors) which should each only be used exactly once.

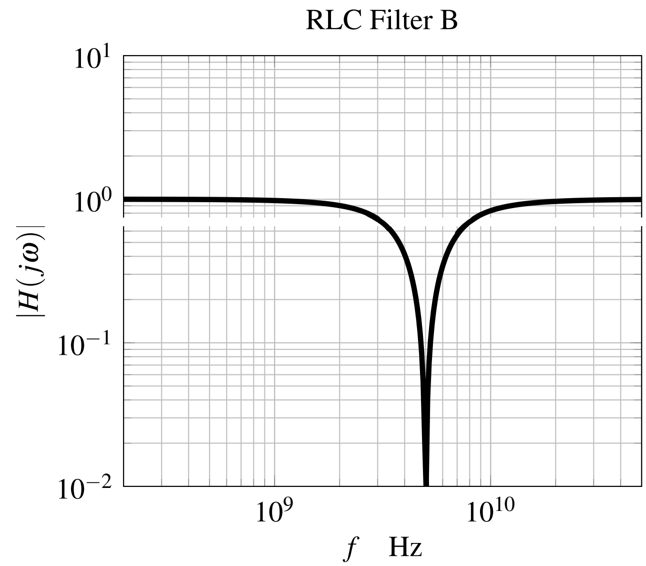
$$C = 66.3\text{pF} \quad C = 31.8\text{pF} \quad L = 31.8\text{pH} \quad L = 66.3\text{pH} \quad R = 1\Omega \quad R = 0.1\Omega$$

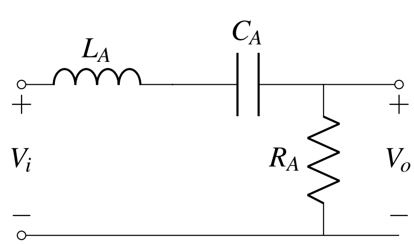
Assign each component to the elements R_A, R_B, C_A, C_B, L_A, L_B in the RLC circuits ('A' and 'B' below), so that the transfer function of each circuit corresponds to its matching Bode plot. Write values next to components. *Hint: the dashed line on the plot is at $\frac{1}{\sqrt{2}}$. It might be useful to think about what its intersections with the main curve represent.*

For your convenience, here are some calculations that may or may not be relevant:

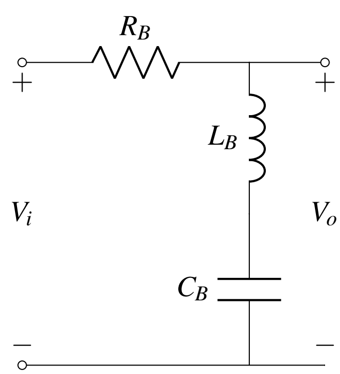
$\frac{2.4\times10^9}{2\pi} = 382\times10^6$	$2.4\times10^9\times2\pi = 15.08\times10^9$	$\frac{5.0\times10^9}{2\pi} = 795\times10^6$	$5.0\times10^9\times2\pi = 31.4\times10^9$
$\frac{1}{382\times10^6} = 2.62\times10^{-9}$	$\frac{1}{15.08\times10^9} = 6.63\times10^{-11}$	$\frac{1}{795\times10^6} = 1.26\times10^{-9}$	$\frac{1}{31.4\times10^9} = 3.18\times10^{-11}$







Circuit A: $H(j\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$



Circuit B: $H(j\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

6. Separation of Variables and Uniqueness (20 pts)

Recall that the classic scalar differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) \tag{1}$$

with initial condition $x(0) = x_0 \neq 0$ has the unique solution $x(T) = x_0 e^{\lambda T}$ for all $T \geq 0$.

(Note: to avoid variable-name confusion here, we are using T as the argument of the solution $x(T)$.)

The separation of variables approach to getting a guess for this problem would proceed as follows:

$$\frac{d}{dt}x(t) = \lambda x(t) \tag{2}$$

$$\frac{dx}{dt} = \lambda x \tag{3}$$

$$\frac{dx}{x} = \lambda dt \text{ separating variables to sides} \tag{4}$$

$$\int_{x_0}^{x(T)} \frac{dx}{x} = \int_0^T \lambda dt \text{ integrating both sides from where they start to where they end up} \tag{5}$$

$$\ln(x(T)) - \ln(x_0) = \lambda T \tag{6}$$

$$\ln(x(T)) = \ln(x_0) + \lambda T \tag{7}$$

$$x(T) = x_0 e^{\lambda T} \text{ exponentiating both sides} \tag{8}$$

and in this case it gave a good guess. Of course, this guess needed to be justified by a uniqueness proof, which you did in the homework.

This exam problem asks you to carry out this program for the time-varying differential equation:

$$\frac{d}{dt}x(t) = \lambda(t)x(t) \tag{9}$$

with initial condition $x(0) = x_0 \neq 0$. You can assume that $\lambda(t)$ is a nice continuously differentiable function of time t that is bounded.

(a) (8 pts) Use the separation of variables approach to get a guess for the solution to the differential equation (9) — namely $\frac{d}{dt}x(t) = \lambda(t)x(t)$ — with initial condition $x(0) = x_0 \neq 0$. **Show work and give a formula for $x(T)$ for $T \geq 0$.**

(HINT: It is fine if your answer involves a definite integral.)

(If you can't solve this for a general $\lambda(t)$, for partial credit, feel free to just consider the special case of $\lambda(t) = -2 - \sin(t)$ and give a guess for that case.)

(You can also get full credit if you follow the approach from discussion section of taking a piecewise-constant approximation and then taking a limit, but that might involve more work.)

(b) (12 pts) **Prove the uniqueness of the solution — i.e. that if any function solves differential equation (9) — namely $\frac{d}{dt}x(t) = \lambda(t)x(t)$ — with the given initial condition $x(0) = x_0 \neq 0$, then it must in fact be the same as your guessed solution everywhere for $T \geq 0$.**

(HINT: A ratio-based argument might be useful. You don't actually need to know the exact form of your guessed solution to carry out much of this argument, but you do need the fact that it is never zero and that it solves (9)*.)*

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

