MATH 54 FINAL EXAM (PRACTICE 1)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ______________________________

GSI's name: ______________________________

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Let $A = \begin{pmatrix} 1 & -1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$. Find a general solution to the homogeneous linear system with coefficient matrix A.

   Solution:

   (b) What is $Nullity(T_A)$? What is $Rank(T_A)$?

   Solution:

2. (25 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \ T\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \ T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Find the standard matrix of T. Is T one-to-one?

Solution:

3. (25 points) Let A be a 4×3 matrix and B be a 3×2 matrix. Show that if the columns of A and B are linearly independent then the columns of AB are linearly independent. Hint: Consider the linear transformations associated to these matrices.

   Solution:

4. (25 points) Let V be a vector space with bases $B = \{\underline{\mathbf{b}}_1, \underline{\mathbf{b}}_2, \underline{\mathbf{b}}_3\}$ and $C = \{\underline{\mathbf{c}}_1, \underline{\mathbf{c}}_2, \underline{\mathbf{c}}_3\}$, where

$$\underline{\mathbf{c}}_1 = \underline{\mathbf{b}}_1 - \underline{\mathbf{b}}_2, \qquad \underline{\mathbf{c}}_2 = \underline{\mathbf{b}}_1 + \underline{\mathbf{b}}_3, \qquad \underline{\mathbf{c}}_3 = \underline{\mathbf{b}}_1 + \underline{\mathbf{b}}_2 + \underline{\mathbf{b}}_3$$

If $(\underline{\mathbf{x}})_B = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, what is $(\underline{\mathbf{x}})_C$?

Solution:

5. (25 points) Give an example of a non-diagonalizable matrix with only real eigenvalues. Carefully justify your answer.

   Solution:

6. (25 points) Compute the minimum distance between $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ and

$$Nul \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 6 & 9 & 12 \\ 2 & 4 & 6 & 8 \end{pmatrix}$$

Solution:

7. (25 points) Perform a singular-value decomposition of the matrix

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

Solution:

Solution (continued) :

8. (25 points) Find a general solution to the following differential equation

$$y'' - 2y' + y = 12t^2e^t$$

Solution:

Solution (continued) :

9. (25 points) Consider the following $\mathbb{R}^3$-valued function on $\mathbb{R}$:

$$\begin{pmatrix} t \\ t \\ t^2 \end{pmatrix}, \quad \begin{pmatrix} t^2 \\ t^2 \\ t^3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix}.$$

Determine if these vector-valued functions are linearly independent. Are they solutions to some 3×3 homogeneous linear system of differential equations? Carefully justify your answers.

Solution:

10. (25 points) Calculate the sine Fourier series of the function $f(x) = x$, on the interval $[0, \pi]$. Use this to prove that

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots = \frac{\pi}{4}$$

Solution:

END OF EXAM