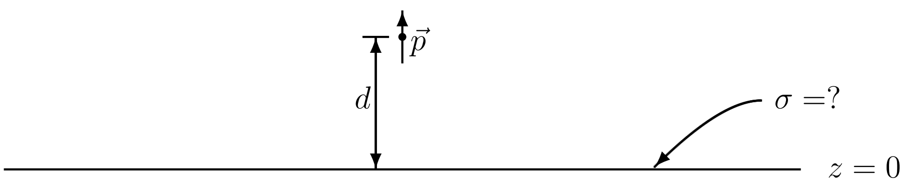
Physics 110A (Electrodynamics) – Midterm Exam / November 4, 2019

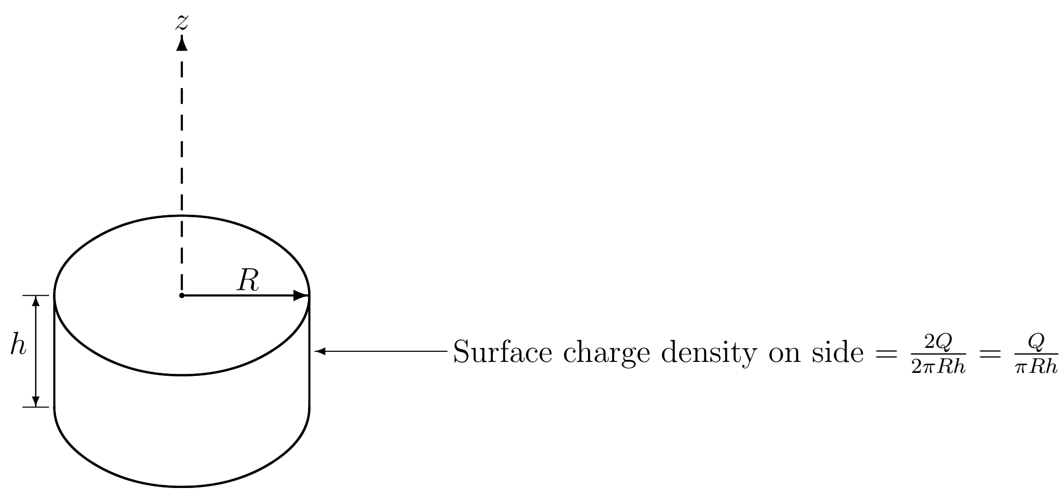
Problem 1 (50 points)



A perfect electric dipole with dipole moment $\vec{p} = p\hat{z}$ is held at height d above an infinite grounded conducting plane. Find the surface charge density σ that is induced on the plane.

Express your result as follows.
If the dipole is at (x, y, z) coordinates $(0, 0, d)$ and the plane is at $z = 0$, express σ as a function of the distance $s = \sqrt{x^2 + y^2}$ from the origin.

Problem 2 (50 points)



A cylinder of height h and radius R is aligned with its axis along the z-axis. A total charge of $-Q$ is uniformly distributed on each of the two bases (with surface charge density $\sigma = -Q/\pi R^2$), and a total charge of $2Q$ is uniformly distributed on the side (with surface charge density $\sigma' = Q/\pi ah$).

Find an approximate expression for the electrostatic potential V produced by the cylinder at large distance r away. Assume $r \gg a, h$ and $V \to 0$ as $r \to \infty$.

Express your result in spherical coordinates with the origin at the center of the cylinder (at height $\frac{h}{2}$, not shown in the picture). $V(r, \theta)$ should fall off as a power law in r.

Note: the cylinder is neutral as a whole, i.e., the total charge is $(-Q) + (-Q) + (2Q) = 0$.

Solution to problem 1

The electric field of a dipole at the origin is

$$\vec{E} = \frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{4\pi\epsilon_0 r^3}$$

We set

$$\hat{r} = \frac{x}{r}\hat{x} + \frac{y}{r}\hat{y} + \frac{z}{r}\hat{z}, \qquad \vec{p}\cdot\hat{r} = \frac{\vec{p}\cdot\vec{r}}{r} = \frac{pz}{r}, \qquad r = \sqrt{x^2+y^2+z^2}.$$

For a dipole at $(0,0,d)$, the electric field at r is

$$\vec{E}_1(\vec{r}) = \frac{1}{4\pi\epsilon_0 r_1^3}\left\{\frac{3p(z-d)}{r_1^2}[x\hat{x} + y\hat{y} + (z-d)\hat{z}] - p\hat{z}\right\}, \qquad r_1 \equiv \sqrt{x^2+y^2+(z-d)^2}$$

We use the method of images and put an image dipole at $(0,0,-d)$. The image dipole has the same dipole moment, because if the original dipole is composed of a $-q$ at $(0,0,d)$ and a $+q$ at $(0,0,d+h)$, with $p = qh$, then the image dipole will be composed of $+q$ at $(0,0,-d)$ and $-q$ at $(0,0,-d-h)$ with dipole moment $(-q)(-h) = qh$.

The image dipole contributes

$$\vec{E}_2(\vec{r}) = \frac{1}{4\pi\epsilon_0 r_2^3}\left\{\frac{3p(z+d)}{r_2^2}[x\hat{x} + y\hat{y} + (z+d)\hat{z}] - p\hat{z}\right\}, \qquad r_2 \equiv \sqrt{x^2+y^2+(z+d)^2}$$

Now, we set

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

and set $z = 0$, and

$$\sigma = \epsilon_0 \vec{E}\cdot\hat{z}.$$

We calculate

$$r_1 = r_2 = \sqrt{s^2+d^2}$$

$$\vec{E}_1(x,y,0)\cdot\hat{z} = \frac{p}{4\pi\epsilon_0 r_1^3}\left(\frac{3d^2}{r_1^2} - 1\right) = \frac{p}{4\pi\epsilon_0 \sqrt{s^2+d^2}^3}\left(\frac{3d^2}{s^2+d^2} - 1\right)$$

and

$$\vec{E}_2(x,y,0)\cdot\hat{z} = \frac{p}{4\pi\epsilon_0 r_2^3}\left(\frac{3d^2}{r_2^2} - 1\right) = \frac{p}{4\pi\epsilon_0 \sqrt{s^2+d^2}^3}\left(\frac{3d^2}{s^2+d^2} - 1\right)$$

So

$$\boxed{\sigma = \frac{p}{2\pi\sqrt{s^2+d^2}^3}\left(\frac{3d^2}{s^2+d^2} - 1\right)}$$

Solution to problem 2

The total monopole and dipole terms of the multipole expansion are zero. The first nonzero multipole is $l = 2$ and we need to calculate

$$q_2 = \int r^2 P_2(\cos\theta)\rho(\vec{r})d\tau = \int r^2 (\frac{3}{2}\cos^2\theta - \frac{1}{2})\rho(\vec{r})d\tau = \frac{1}{2}\int (2z^2 - x^2 - y^2)\rho d\tau$$

Bases

For the top base we set

$$(x, y, z) = (s\cos\phi, s\sin\phi, \frac{h}{2}), \qquad \rho d\tau \to -\frac{Q}{\pi R^2} s ds d\phi.$$

Then,

$$\begin{aligned}
\frac{1}{2}\int (2z^2 - x^2 - y^2)\rho d\tau &= -\frac{Q}{2\pi R^2}\int_0^{2\pi}\int_0^R [2(\frac{h}{2})^2 - s^2] s ds d\phi = -\frac{Q}{R^2}\int_0^R [2(\frac{h}{2})^2 - s^2] s ds \\
&= -\frac{Q}{R^2}(\frac{h^2 R^2}{4} - \frac{R^4}{4}) = \frac{Q}{4}(R^2 - h^2)
\end{aligned}$$

The contribution of the bottom base is the same.

Side

For the side we set

$$(x, y, z) = (R\cos\phi, R\sin\phi, z), \qquad \rho d\tau \to \frac{Q}{\pi R h} R d\phi dz = \frac{Q}{\pi h} d\phi dz.$$

Then,

$$\begin{aligned}
\frac{1}{2}\int (2z^2 - x^2 - y^2)\rho d\tau &= \frac{Q}{2\pi h}\int_0^{2\pi}\int_{-\frac{h}{2}}^{\frac{h}{2}} [2z^2 - R^2] dz d\phi = \frac{Q}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} [2z^2 - R^2] dz \\
&= \frac{Q}{h}[\frac{4}{3}(\frac{h}{2})^3 - R^2 h] = Q(\frac{h^2}{6} - R^2)
\end{aligned}$$

Altogether, we get

$$q_2 = 2\frac{Q}{4}(R^2 - h^2) + Q(\frac{h^2}{6} - R^2) = -(\frac{h^2}{3} + \frac{R^2}{2})Q$$

The potential is

$$V \approx \frac{1}{4\pi\epsilon_0}\frac{q_2}{r^3}P_2(\cos\theta) = \frac{q_2}{8\pi\epsilon_0}\left(\frac{3\cos^2\theta - 1}{r^3}\right).$$