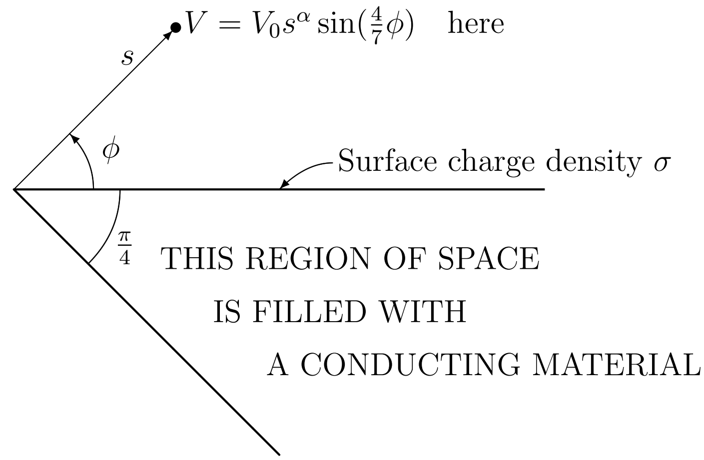
Physics 110A (Electrodynamics) – Midterm Exam / October 7, 2019

Problem 1 (50 points)



An infinite conducting wedge has an opening angle of $\pi/4$.
The conducting wedge occupies the portion of space given, in cylindrical coordinates, by

$$-\infty < z < \infty, \qquad 0 \le s < \infty, \qquad \frac{7\pi}{4} \le \phi \le 2\pi.$$

The remaining portion of space ($0 < \phi < \frac{7\pi}{4}$) has an electric field with a scalar potential given by

$$V = V_0\, s^\alpha \sin(\frac{4}{7}\phi) \qquad \text{for some constants } V_0 \text{ and } \alpha > 0.$$

(a) If the space outside the conductor is free of charges, what is the value of α? **(25 points)**

(b) Find the surface charge density σ on the conductor surface at $\phi = 0$.
The answer should be a function of s and the unknown constants V_0 and α. **(25 points)**

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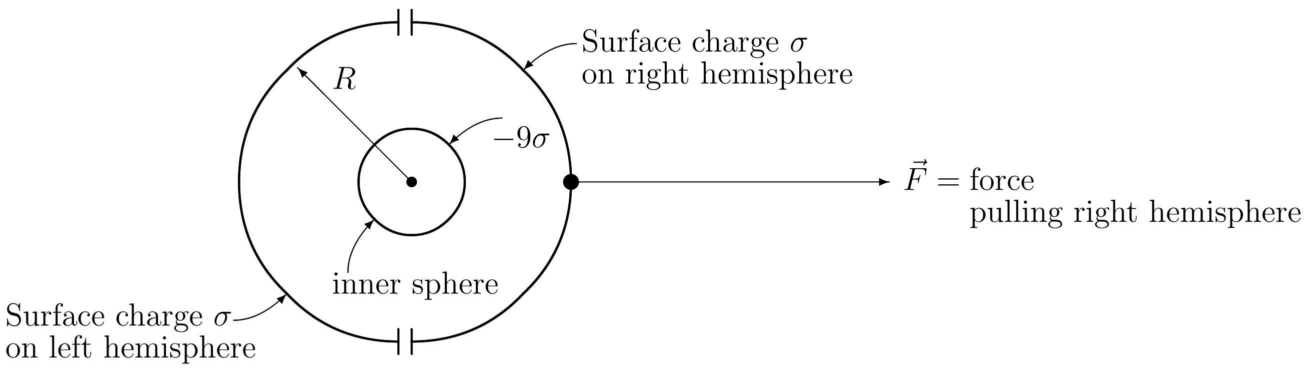
Problem 2 (50 points)



A sphere of radius R is carrying a uniform surface charge density σ. Inside it is an smaller sphere of radius $R/3$ carrying (an oppositely charged) surface charge density -9σ. The outer sphere is cut in half, and the two hemispheres (left and right) are separated, with a very (infinitesimally) thin gap between them.

(a) Calculate the electrostatic energy of this configuration (ignoring the thin gap). **(25 points)**

(b) Calculate the magnitude $|\vec{F}|$ of the force $\vec{F}$ needed to keep the right hemisphere apart from the left one. ($\vec{F}$ must balance the force exerted on the right hemisphere by the inner sphere and the left hemisphere.) **(25 points)**

Solution to problem 1

(a) This problem only requires the equation $\nabla^2 V = -\frac{\rho}{\epsilon_0}$ which becomes $\nabla^2 V = 0$ in a region empty of charges. We calculate (in the range $0 < \phi < \frac{7}{4}\phi$):

$$\begin{aligned} \nabla^2 V &= \frac{1}{s}\frac{\partial}{\partial s}(s\frac{\partial V}{\partial s}) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \\ &= \frac{1}{s}\frac{\partial}{\partial s}(\alpha V_0 s^\alpha \sin(\frac{4}{7}\phi)) - (\frac{4}{7})^2\frac{1}{s^2}V_0 s^\alpha \sin(\frac{4}{7}\phi) = \alpha^2 V_0 s^{\alpha-2}\sin(\frac{4}{7}\phi) - (\frac{4}{7})^2 V_0 s^{\alpha-2}\sin(\frac{4}{7}\phi) \end{aligned}$$

So

$$0 = \nabla^2 V \Longrightarrow \alpha^2 = (\frac{4}{7})^2 \Longrightarrow \alpha = \frac{4}{7}.$$

(b) This problem only requires the equation $\sigma = \epsilon_0 \vec{E}\cdot\hat{n}$ together with $\vec{E} = -\vec{\nabla}V$ and $\hat{n} = \hat{\phi}$. The details:

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial s}\hat{s} - \frac{1}{s}\frac{\partial V}{\partial \phi}\hat{\phi} - \frac{\partial V}{\partial z}\hat{z} = -\alpha V_0 s^{\alpha-1}\sin(\frac{4}{7}\phi)\hat{s} - \frac{4}{7}V_0 s^{\alpha-1}\cos(\frac{4}{7}\phi)\hat{\phi} \xrightarrow{\phi=0} -\frac{4}{7}V_0 s^{\alpha-1}\hat{\phi}$$

So

$$\sigma = \epsilon_0 \vec{E}\cdot\hat{n} = \epsilon_0\vec{E}\cdot\hat{\phi} = -\frac{4}{7}\epsilon_0 V_0 s^{\alpha-1}$$

Solution to problem 2

(a) There are several ways to solve this problem. The first requires the equation $W = \frac{1}{2}\epsilon_0\int E^2 d\tau$ and we have to calculate $\vec{E}$ using Gauss's law. We have to remember that σ is a *surface charge*, and we get the total charge of each sphere by multiplying by the area of the sphere, $4\pi R^2$, or $4\pi(\frac{R}{3})^2$. We also have to note that the electric field inside the inner sphere is 0 (Gauss's Law), and it turns out that the electric field outside the outer sphere is also zero (again, by Gauss's Law).

Here are the details. We choose spherical coordinates so that the direction of $\vec{F}$ is the $\hat{z}$ ($\theta = 0$) axis. The charges on the outer and inner conductors are

$$Q_{\text{outer}} = 4\pi R^2\sigma, \qquad Q_{\text{inner}} = 4\pi(\frac{R}{3})^2(-9\sigma) = -Q_{\text{outer}}.$$

So the total charge is 0 and by Gauss's law, the electric field for $r > R$ is $\vec{E} = 0$. Also, by Gauss's law, the electric field for $\frac{R}{3} < r < R$ is given by

$$\vec{E} = \frac{Q_{\text{inner}}\hat{r}}{4\pi\epsilon_0 r^2} = -\frac{4\pi R^2\sigma\hat{r}}{4\pi\epsilon_0 r^2} = -\frac{R^2\sigma\hat{r}}{\epsilon_0 r^2}$$

$$\vec{E} = \begin{cases} 0 & \text{for } 0 \le r < \frac{R}{3} \\ -\frac{R^2\sigma\hat{r}}{\epsilon_0 r^2} & \text{for } \frac{R}{3} < r < R \\ 0 & \text{for } R < r < \infty \end{cases}$$

We can use either $W = \frac{1}{2}\int \rho V d\tau$ or

$$\begin{aligned}
W &= \frac{1}{2}\epsilon_0 \int E^2 d\tau = \frac{1}{2}\epsilon_0 \int_0^{2\pi}\int_0^{\pi}\int_{\frac{R}{3}}^{R} \left(-\frac{R^2\sigma\hat{r}}{\epsilon_0 r^2}\right)^2 r^2 dr \sin\theta d\theta d\phi \\
&= \int_0^{2\pi}\int_0^{\pi}\int_{\frac{R}{3}}^{R} \frac{R^4\sigma^2}{2\epsilon_0 r^2}\sin\theta dr d\theta d\phi = 4\pi\int_{\frac{R}{3}}^{R}\frac{R^4\sigma^2 dr}{2\epsilon_0 r^2} = \frac{4\pi R^4\sigma^2}{2\epsilon_0}\left(\frac{1}{R/3}-\frac{1}{R}\right) = \frac{4\pi R^3\sigma^2}{\epsilon_0}
\end{aligned}$$

If, instead, we choose to use the formula $W = \frac{1}{2}\int \rho V d\tau$, we can note that V is constant on each of the conductors and so

$$W = \frac{1}{2}\int \rho V d\tau = \frac{1}{2}V_{\text{inner}}Q_{\text{inner}} + \frac{1}{2}V_{\text{outer}}Q_{\text{outer}} = \frac{1}{2}(V_{\text{outer}} - V_{\text{inner}})(4\pi R^2\sigma)$$

and to complete the problem, we need to calculate the voltage difference $V_{\text{outer}} - V_{\text{inner}}$. This is done the same way as in the next method, which is treating the system as a capacitor (see below).

Alternatively, we can look at this configuration as a capacitor and calculate its voltage as

$$V = -\int_{R/3}^{R} E_r dr = \frac{R^2\sigma}{\epsilon_0}\int_{R/3}^{R}\frac{dr}{r^2} = \frac{2R\sigma}{\epsilon_0}$$

and then

$$W = \frac{1}{2}QV = \frac{1}{2}Q_{\text{outer}}V = \frac{4\pi R^3\sigma^2}{\epsilon_0}.$$

(b) This problem requires the formula $\vec{f} = \frac{1}{2}\sigma\vec{E}_{\text{average}}$, and integration over parts of a sphere, in spherical coordinates. Here are the details.

The force per unit area is

$$\vec{f} = \frac{1}{2}\sigma\vec{E}_{\text{average}}\bigg|_{r=R} = \frac{1}{2}\sigma\left(-\frac{R^2\sigma\hat{r}}{\epsilon_0 R^2}\right) = -\frac{\sigma^2\hat{r}}{2\epsilon_0}$$

We'll choose the z axis in such a way that the cut is along the equator. Thus, by symmetry, we only need the $\hat{z}$-component of the force, since the $\hat{x}$ and $\hat{y}$ components are 0. We have

$$f_z = -\frac{\sigma^2}{2\epsilon_0}\cos\theta$$

and the total z component is

$$F_z = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} f_z R^2\sin\theta d\theta d\phi = 2\pi R^2\int_0^{\frac{\pi}{2}}\left(-\frac{\sigma^2}{2\epsilon_0}\cos\theta\right)\sin\theta d\theta = -\frac{\pi\sigma^2R^2}{2\epsilon_0}[\sin^2\theta]_0^{\pi/2} = -\frac{\pi\sigma^2R^2}{2\epsilon_0}$$

So, the answer is

$$|\vec{F}| = \frac{\pi\sigma^2R^2}{2\epsilon_0}.$$

Alternatively, we can take the x-axis in the direction of the force. Then,

$$f_x = -\frac{\sigma^2}{2\epsilon_0}\sin\theta\cos\phi$$

and the total x component is

$$F_x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi} f_x R^2 \sin\theta d\theta d\phi = -\frac{\sigma^2 R^2}{2\epsilon_0}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\phi d\phi\int_0^{\pi}\sin^2\theta d\theta$$

and we get the same answer as above if we set

$$\int_0^{\pi}\sin^2\theta d\theta = \int_0^{\pi}(\frac{1-\cos 2\theta}{2})d\theta = \left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_0^{\pi} = \frac{\pi}{2}, \qquad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\phi d\phi = 2.$$

Another method we can consider is to change the radius of the outer sphere to $R+u$, while keeping the total charge fixed to $4\pi R^2\sigma$, and then calculate W as a function of u and take the derivative $-dW/du$. This is a good idea, that certainly deserves partial credit, but it will give us the *pressure*, which is a scalar, and we need the force, which is a vector.

Let us proceed anyway, and then we'll finish the problem with a small trick. We can also take a shortcut in calculating dW/du. Since Q_{outer} doesn't change as we change $R \to R+u$, we can calculate the change in energy as

$$\delta W = \delta\left(\frac{1}{2}\rho V d\tau\right) = \frac{1}{2}Q_{\text{outer}}\delta V_{\text{outer}} = \frac{1}{2}Q_{\text{outer}}\int_R^{R+u} E_r dr$$

and therefore

$$-\frac{dW}{du} = \frac{1}{2}Q_{\text{outer}}E_r\Big|_{\text{at } r=R, \text{ just inside the ball}} = \frac{1}{2}(4\pi R^2\sigma)\left(-\frac{\sigma}{\epsilon_0}\right) = -\frac{2\pi R^2\sigma^2}{\epsilon_0}$$

We get the pressure by dividing by the total surface area. (Recall from thermodynamics that $dW = -PdV$, and $dV = 4\pi R^2 du$ here.)

$$P = \frac{1}{4\pi R^2}\left(-\frac{2\pi R^2\sigma^2}{\epsilon_0}\right) = -\frac{\sigma^2}{2\epsilon_0}$$

So far this is similar to what we found above. To proceed, we can either integrate the $\hat{z}$ component as above, or we can take an analogy from hydrodynamics: if we have a balloon of radius R filled with a gas at equal pressure P, the force on one hemisphere is equal to $\pi R^2 P$. So in our case the answer should be

$$|\vec{F}| = \frac{\sigma^2}{2\epsilon_0}(\pi R^2) = -\frac{\pi R^2\sigma^2}{2\epsilon_0}.$$

All of this is, of course, not the way you were expected to solve the problem, but if you suggested calculating dW/dR, you got partial credit.