**Mathematics 53. Fall Semester 2018** **Professor: Daniel Tataru**

**Final Exam**

**Dec, 12, 2018, 3:00-6:00**

Your Name: ______________________ (write clearly !)

Your ID: ______________________

TA's Name: ______________________

Section time: ______________________

**Directions:** This is a *closed* book exam. No calculators, cell phones, tablets, laptops and other electronic devices are allowed.

**Remember:** Answers without explanations will not count. You should **show your work**. Solve each problem on its own page. If you need extra space you can use the backs of the pages. The scratch paper is only for your in class use and will not be graded.

1. a) (5) For a function $f(x, y)$ in $\mathbb{R}^2$ define its Laplacian $\Delta f$ (also denoted in the book by $\nabla^2 f$).

b) (10) Let $g$ be a twice differentiable function of one variable and $f(x, y) = g(r)$ where $r$ is the radius in polar coordinates. Compute the expression $\Delta f$ in terms of $g$ and its derivatives.

c) (10) Check whether the function $f(x, y) = \ln(x^2 + y^2)$ solves the Laplace equation $\Delta f = 0$.

2. An asteroid is described by the equation

$$x^2 + 2y^2 + 3z^2 \leq 1.$$

a) (5) What is the shape of the asteroid ? Sketch it.

b) What is the equation for the tangent plane at some point $P = (x_0, y_0, z_0)$ on the surface ?

c) (10) What are the points on its surface which are visible from a spaceship located at $S = (1, 1, 1)$?

3. Consider the curve $\gamma$ which is defined as the portion of the intersection of the hyperboloid $H = \{x^2 + y^2 - z^2 = 1\}$ with the cylinder $C = \{y = x^2\}$ between the points $P = (-1, 1, 1)$ and $Q = (1, 1, 1)$.

a) Find the direction of the tangent vector to $\gamma$ at the point $Q$.

b) Evaluate the integral

$$I = \int_\gamma F \cdot dr, \qquad F = \left(\frac{1}{1+y+z}, -\frac{x}{(1+y+z)^2}, 2z - \frac{x}{(1+y+z)^2}\right)$$

4. Find the maximum of the function

$$f(x, y, z) = xyz + x + y + z$$

within the sphere $S = \{x^2 + y^2 + z^2 \leq 1\}$.

5. Evaluate the following integrals:

a)

$$\int_0^1 \int_0^1 \sin(\max\{x^2, y^2\})\, dx\, dy$$

b)

$$\int_0^1 \int_0^{\sqrt{1-z^2}} \int_z^{\sqrt{1-y^2}} x^2 y\, dx\, dy\, dz$$

6. Let $D$ be the solid bounded from above by the sphere $x^2 + y^2 + z^2 = 2$ and from below by the paraboloid $z = x^2 + y^2$. Find its volume.

7. a) (10) Carefully state Stokes's Theorem.

b) (15) The curve $C$ is the intersection of the cylinder $x^2 + y^2 = 1$ with the hyperbolic paraboloid

$$z = x^2 - y^2$$

taken with the counterclockwise orientation. Evaluate the line integral

$$I = \oint_C F \cdot d\mathbf{r}, \qquad F = (x - y, x + y, z^7)$$

8. Consider a pizza given by the planar domain

$$D = \{4x^2 + y^2 \le 4y\}$$

and with density $\rho(x, y) = 4y - y^2$. Find its mass.

9. Evaluate the integral

$$\iint_S z^{2018}\, dS, \qquad S = \{x^2 + y^2 + z^2 = 1\}$$