**Physics 7B Midterm 2 Solutions - Fall 2019**
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# Problem 1

1. (5 pt) Three infinite flat sheets of charge.

(2 pt) For a single infinite flat sheet of charge, can either write or derive the magnitude from an electric field using Gauss' Law.

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{encl}}{\epsilon_0}$$
$$E(2A) = \frac{\sigma A}{\epsilon_0}$$
$$E = \frac{\sigma}{2\epsilon_0}$$

(2 pt) With three sheets, superposition is applied so that the electric field from each sheet contributes to a region.

(1 pt) Therefore, for each region $E_1 = -\frac{s}{2\epsilon_0}$, $E_2 = \frac{s}{2\epsilon_0}$, $E_3 = \frac{3s}{2\epsilon_0}$, and $E_4 = \frac{s}{2\epsilon_0}$ pointing to the right.

2. (5 pt) Proton fired toward a charged sheet.

i. (2 pt) Force on a charge in a electric field is equal to $\vec{F} = q\vec{E}$, so the force on the charge next to the infinite sheet is $\vec{F} = \frac{q\sigma}{2\epsilon_0}$.

ii. (3 pt) One can calculate the distance away from the sheet either by using conservation of energy or by calculating energy from force. With conservation of energy, one solve as such.

$$K = W_{stop} = q\Delta V = qEd$$
$$d = \frac{K}{qE} = \frac{2K\epsilon_0}{q\sigma}$$

Alternatively, with force, one can solve for the initial velocity and the distance it takes for that velocity to go to zero starting from $K = \frac{1}{2}mv^2$ and $v = \sqrt{\frac{2K}{m}}$.

$$F = ma = \frac{q\sigma}{2\epsilon_0}$$
$$a = \frac{q\sigma}{2m\epsilon_0}$$
$$\Delta t = \frac{v}{a} = \frac{2m\epsilon_0}{q\sigma}\sqrt{\frac{2K}{m}}$$
$$d = \frac{1}{2}a\Delta t^2 = \frac{v^2}{2a} = \frac{K}{m}\frac{2m\epsilon_0}{q\sigma} = \frac{2K\epsilon_0}{q\sigma}$$

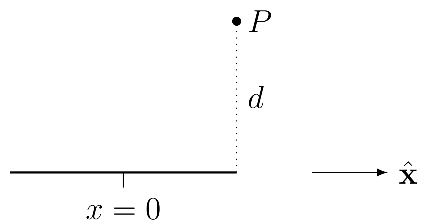
3. (5 pt) Equipotential lines.

i. (1 pt) pt) We know that $W = q\Delta V$, therefore the order magnitude wise is (2 pt) $W_{AB} = W_{CD} > W_{AC}$.

ii. (2 pt) We know that electric field is dependent on the density of the equipotential lines, so that the closer together they are, the stronger the field. From this, we can see that $E_A > E_C > E_B > E_D$. $E_A > E_B > E_C > E_D$ is also acceptable.

# Problem 2

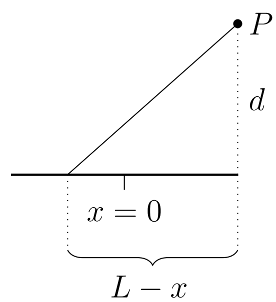
(a) The bar extends from $x = -L$ to $x = L$ and has linear charge density $\lambda = ax$ $(a > 0)$.




If we set the potential equal to zero at spatial infinity, then the potential at point $P$ is given by

$$V = \int \mathrm{d}V = \frac{1}{4\pi\epsilon_0} \int \frac{\mathrm{d}q}{\ },$$

where  is the separation distance between a point $\mathrm{d}q$ on the rod and the point $P$.




We write $\mathrm{d}q = \lambda \mathrm{d}x$ and, using the diagram above, $= \sqrt{d^2 + (L - x)^2}$, so

$$V = \frac{a}{4\pi\epsilon_0} \int_{-L}^{L} \frac{x\mathrm{d}x}{\sqrt{d^2 + (L - x)^2}}.$$

In order to use the integrals provided on the equation sheet, we need to change variables. Define $u = L - x$, so that $\mathrm{d}u = -\mathrm{d}x$, and adjust the limits of integration:

$$V = \frac{a}{4\pi\epsilon_0} \int_{2L}^{0} \frac{-(L - u)\mathrm{d}u}{\sqrt{d^2 + u^2}} = \frac{a}{4\pi\epsilon_0} \int_{0}^{2L} \left( \frac{L\mathrm{d}u}{\sqrt{d^2 + u^2}} - \frac{u\mathrm{d}u}{\sqrt{d^2 + u^2}} \right).$$

Referring to the equation sheet, we have

$$V = \frac{a}{4\pi\epsilon_0}\left[L\ln\left(u+\sqrt{d^2+u^2}\right)\Big|_0^{2L} - \sqrt{d^2+u^2}\Big|_0^{2L}\right]$$

or

$$V = \frac{a}{4\pi\epsilon_0}\left[L\ln\left(\frac{2L+\sqrt{d^2+4L^2}}{d}\right) - \sqrt{d^2+4L^2}+d\right].$$

Let's check units: because $\lambda = ax$ had dimensions of charge-per-length, $a$ must have dimensions of charge-per-length$^2$. The quantity in square brackets overall has dimensions of length, and thus our expression does have the correct units of potential.

(b) Because the rod has a negatively charged half and a positively charged half, it has a nonzero dipole moment. If an electric field is turned on, the rod will accordingly behave like a dipole: it will tend to rotate to its equilibrium position, making an angle of $0°$ relative to the electric field. (Note that the problem statement asks for the equilibrium angle *relative to the electric field.*)

# Rubric

Part (a):

- 15 points: Correct
- 12 points: The integral for $V$ is correctly set up but incorrectly or incompletely evaluated
- 10 points: The integral for $V$ is set up with minor errors (e.g. the limits of integration describe the length of the rod but are incorrect, the separation distance involves $L+x$ instead of $L-x$, etc)
- 8 points: The integral for $V$ is set up with more significant errors (e.g. the limits of integration do not describe the length of the rod, the separation distance has the incorrect functional form, etc)
- 3 points: Some attempt was made to set up an expression for $V$ (e.g. there is no integration, an incorrect formula for $V$ is used, the relationship between $V$ and $\mathbf{E}$ is incorrectly applied, etc)
- +1 point: Units were checked
- −1 point: Fundamental errors (e.g. $V$ is treated as a vector quantity)

Part (b)

- 5 points: Correct
- 2 points: The equilibrium angle is not correctly identified, but other relevant work is shown (e.g. recognizing that the rod behaves like an electric dipole and will rotate)

# Problem 3

## 1 Solution

a) Using Gauss's law, the electric field do to a sphere of charge Q is

$$\vec{E}(r) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}.$$

Really close to the surface of the sphere, we have $r \approx R$, so we will have

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2}\hat{r}.$$

An alternate solution uses the fact that the electric field will look like $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}$ right outside of a conductor, where $\sigma$ is the surface charge density and $\hat{n}$ is the direction normal to the surface.

b) Take the system as a sphere of surface charge density $\sigma$ plus a disk of surface charge density $-\sigma$. At point A, the disk looks like an infinite plane. Therefore,

$$\vec{E}_{disk} = -\frac{\sigma}{2\epsilon_0}\hat{r},$$

where $\sigma = \frac{Q}{4\pi R^2}$.

Using superposition,

$$\vec{E} = \vec{E}_{sphere} + \vec{E}_{disk} = \frac{Q}{8\pi\epsilon_0 R^2}\hat{r}.$$

c) Conservation of energy tells us that $\Delta U + \Delta\text{KE} = 0$. Taking the electric potential to be 0 at infinity, we have $\Delta U = qV(0)$.

Using the electric field outside of the conductor and the fact that $\vec{E} = 0$ inside the shell, $V(0) = \frac{Q}{4\pi\epsilon_0 R}$. Therefore,

$$\frac{1}{2}mv^2 = q\frac{Q}{4\pi\epsilon_0 R}$$

$$\Rightarrow v = \sqrt{\frac{qQ}{2\pi\epsilon_0 Rm}}$$

# Rubric

Part A

- (1 point) Sets up Gauss's Law for the sphere
- (1 point) Correctly finds surface area of Gaussian surface

- (1 point) Uses $r = R$
- (1 point) Finds correct magnitude of electric field
- (1 point) Includes correct direction of electric field

Part B

- (2 points) Describes hole as a plane with charge density $-\sigma$
- (4 points) Correctly finds the electric field due to the disk
- (2 points) Uses superposition to find the total electric field
- (1 point) Finds correct magnitude of electric field
- (1 point) Includes correct direction of electric field

Part C

- (1 point) Sets up conservation of energy, $\Delta U + \delta \mathrm{KE} = 0$
- (1 point) Uses $U = q\Delta V$
- (2 points) Correctly calculates $\Delta V$ from $\vec{E}$ due to the sphere
- (1 point) Correctly solves for $v$

# Problem 4

# Solution

## (a)

First, applying Gauss's law to figure out E(r) inside the capacitor: choosing a cylinder Gauss surface with radius $r$ and length $h$ [2PT]

$$\oint_S \vec{E} \cdot d\vec{A} = 2\pi r h E(r) = Q/\epsilon_0 \quad \left( E(r) = \frac{Q}{2\pi\epsilon_0 h r} \right)$$

Then we compute the potential difference between two shells [2PT]

$$V = \int_a^b E(r) dr \quad \left( = \frac{Q}{2\pi\epsilon_0 h} \ln \frac{b}{a} \right)$$

Finally, we get the capacitance formula [(2+1)PT]:

$$C = \frac{Q}{V} \Rightarrow \frac{2\pi\epsilon_0 h}{\ln(b/a)}$$

## (b)

The energy stored in the capacitor is [(1+1)PT]

$$U = \int_0^Q V(q)dq = \frac{1}{C}\int_0^Q qdq \Rightarrow \frac{Q^2}{2C}$$

Inputting the value from (a), we get [2PT]

$$U = \frac{Q^2 \ln(b/a)}{4\pi\epsilon_0 h}$$

## (c)

The new system can be regarded as two capacitor connected in parallel, so the equivalent capacitance is [(1+1)PT]

$$\text{"Parallel"} \Rightarrow C(x) = C_1 + C_2$$

*(P.S. Treating x=d will just lose 2 pts here. Consequent steps not affected.)*

and each capacitance is [2PT]

$$C_1 = \frac{2\pi\epsilon_1\epsilon_0 x}{\ln(b/a)}; \quad C_2 = \frac{2\pi\epsilon_0(h-x)}{\ln(b/a)}$$

*(P.S. Using $\epsilon_1$ instead of $\epsilon_1\epsilon_0$ is also acceptable with no penalty.)*

Therefore [1PT]

$$C(x) = \frac{2\pi\epsilon_0(h+(\epsilon_1-1)x)}{\ln(b/a)}$$

## (d)

The force $F(x)$ pulls the dielectric in at the cost of capacitor's energy $U(x)$, so [2PT]

$$F(x) = -\frac{dU(x)}{dx}$$

*(P.S. Using F=U/d only get 1 pt.)*

Inputting the value from (c), energy stored in the capacitor is [(1+1)PT]

$$U(x) = \frac{Q^2}{2C(x)} \Rightarrow \frac{Q^2\ln(b/a)}{4\pi\epsilon_0(h+(\epsilon_1-1)x)}$$

So the force is [2PT]

$$F(x) = \frac{Q^2\ln(b/a)}{4\pi\epsilon_0}\frac{\epsilon_1-1}{(h+(\epsilon_1-1)x)^2}$$

# Problem 5

Two identical rods of length $L$ lie on the $x$-axis and carry uniform charge $+Q$ and $-Q$, as shown below.

**(a)** (12 pts.) Find an expression for the electric field strength as a function of position $x$ for points to the right of the right hand rod. We can find the electric field simply by integrating

Coulomb's law.

$$\mathbf{E}(x) = \int \frac{dq\hat{r}}{4\pi\epsilon_0 r^2}$$

We will consider a point lying at a position $x$ on the $x$-axis and integrate over charge $dq = \pm Q dx'/L$ where $x'$ will be our integration variable (the location of the charges in Coulomb's law). All electric field contributions lie in the $\hat{x}$ directions. We break the integral over all charges into a sum to split up $+Q$ and $-Q$.

$$\mathbf{E}(x) = \int_0^L \frac{Q dx' \hat{x}}{4\pi\epsilon_0 L(x-x')^2} + \int_{-L}^0 \frac{-Q dx' \hat{x}}{4\pi\epsilon_0 L(x-x')^2} = \frac{Q\hat{x}}{4\pi\epsilon_0 L}\left(\int_0^L \frac{dx'}{(x'-x)^2} - \int_{-L}^0 \frac{dx'}{(x'-x)^2}\right)$$

These integrals are easy to compute after a substitution $u = (x' - x)$, we get $\int u^{-2} du = -u^{-1} = (x - x')^{-1}$. Therefore

$$\mathbf{E}(x) = \frac{Q\hat{x}}{4\pi\epsilon_0 L}\left(\frac{1}{x-L} - \frac{1}{x} - \frac{1}{x} + \frac{1}{x+L}\right)$$

This is an acceptable form, but to ease later calculations, I will simplify it. Also, the problem only asks us about strength, so I'll suppress vector notation now. $\mathbf{E}(x) \to E(x)$.

$$\boxed{E(x) = \frac{Q}{4\pi\epsilon_0 L}\frac{x(x-L) + x(x+L) - 2(x^2 - L^2)}{x(x^2 - L^2)} = \frac{QL}{2\pi\epsilon_0 x(x^2 - L^2)}}$$

Rubric: 2 pts. for writing Coulomb's law
2 pts. for writing a continuous version of Coulomb's law
3 pts. for the right substitutions in to the integral
3 pts. for the separation into two integrals and the right bounds
2 pts. for correct integration leading to the correct result

**(b)** (5 pts.) Show that your result has the $1/x^3$ dependence of a dipole field for $x \gg L$. With our simplified form, we can see that if $x \gg L$, then $x^2 - L^2 \approx x^2$, giving us the desired $x$

dependence. Let's be more rigorous and use a method that works with or without simplifying our $E(x)$. Write things in terms of a small parameter $\delta = L/x$ and perform a Taylor series expansion.

$$E(x) = \frac{Q\delta}{2\pi\epsilon_0 x^2(1-\delta^2)} = \frac{Q}{2\pi\epsilon_0 x^2}\delta(1-\delta^2)^{-1}$$

We can now perform a Taylor expansion (i.e. use the binomial expansion) on $(1-\delta^2)^{-1} = 1 + \delta^2 + \delta^4 + ....$ Taking only the first term gives us the asymptotic $x$ dependence

$$\boxed{E(x) \approx \frac{QL}{2\pi\epsilon_0 x^3}}$$

Rubric: 3 pts. for a correct argument about what happens when $x \gg L$ or for attempting a Taylor Series
2 pts. for the correct result in the limit $\delta \to 0$

**(c)** (3 pts.) What is the dipole moment of this configuration? There are 2 basic approaches to this problem - we can use the given formula for the potential form a dipole and try to match the dipole moment to our equation for $E(x)$, or we can use the formula for the dipole moment and perform an integral over the charge distribution. I'll do both.

For a dipole we are given $V = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$. We can find an electric field from this potential by taking a derivative, but we're more familiar with the other direction. Let's find the potential from our rod by integrating the electric field.

$$V(x) = -\int_\infty^x \frac{QL}{2\pi\epsilon_0 x'^3}dx' = \frac{QL}{4\pi\epsilon_0 x^2}$$

Setting this equal to the given expression for $V$ and noting that on the $x$ axis $\cos\theta = 1$, we get

$$\boxed{p = QL}$$

.

The alternative is think of the rod as the superposition of many dipoles with separation $2x$ and charge $Qdx/L$. This is the same as using the dipole formula for a generic charge distribution $p = \sum_i q_i \mathbf{x}_i$. We get

$$p = \int dp = \int d(x)dq = \int_0^L \frac{2Qx}{L}dx = QL$$

The answers agree.

Rubric (method 1): 1 pt. for writing the potential of a dipole
1 pt. for comparing $E(x)$ to that potential in the limit $x \gg L$
1 pt. for the correct result

Rubric (method 2): 1 pt. for writing the formula for dipole moment
1 pt. for integrating the charge distribution to find the total dipole moment
1 pt. for the correct result

**(d)** (3 pts.) How would the behavior of the field change in the limit $x \gg L$ if the two rods were replaced by a single rod of length $2L$ and charge $+Q$? If we instead have a single rod with charge $Q$, then far away form the rod we will see a point charge (a monopole) and the electric field will look like that of a monopole.

$$E(x) = \frac{Q}{4\pi\epsilon_0 x^2}$$

Rubric: 2 pts. for saying it looks like a point charge
1 pt. for providing either the full form of $E(x)$ or saying $E(x) \propto x^{-2}$