1. (2 points each) Determine if the following sequences converge. Write a short justification.

(1) $\sin(n\pi/12)$, $n \to \infty$.

no. It is periodic in n, $a_n = a_{n+24}$.

(2) $\frac{1}{n}\sin(n\pi/12)$, $n \to \infty$.

yes. it convergences to 0, since

$$\left|\frac{1}{n}\sin(n\pi/12)\right| < \left|\frac{1}{n}\right| \cdot \left|\sin\left(\frac{\pi n}{12}\right)\right| < \frac{1}{n}.$$

(3) $\sqrt{n+1} - \sqrt{n}$, $n \to \infty$

converges to 0, since

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0$$

(4) $n^{\ln n}/e^n$, $n \to \infty$ converges.

$$e^{(\ln n)^2}/e^n = e^{(\ln n)^2 - n}$$

since $(\ln n)^2 - n \to -\infty$ as $n \to \infty$, $e^{(\ln n)^2 - n} \to 0$

(5) $\frac{(n+\sin n)^2}{(n+\cos n)^2}$, $n \to \infty$

converges to 1.

$$\left(\frac{n+\sin n}{n+\cos n}\right)^2 = \left(\frac{1+\frac{1}{n}\sin n}{1+\frac{1}{n}\cdot\cos n}\right)^2 \to 1.$$

2. (2 points each) Determine if the following series converge. Write a short justification.

(1) $\sum_{n=1}^{\infty} e^{-n} n^2$

Converges. by ratio test

$$\rho = \lim_{n\to\infty} \left| \frac{e^{-(n+1)} (n+1)^2}{e^{-n} \cdot n^2} \right| = \frac{1}{e} < 1$$

(2) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

converges, by alternating series test. since

$\frac{1}{\sqrt{n}}$ is decreasing

(3) $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{(n+3)(n+4)(n+5)}$

Does not converges. We compaire it with $\sum_{n=1}^{\infty} \frac{1}{n}$.

~~since~~ if $a_n = \frac{(n+1)(n+2)}{(n+3)(n+4)(n+5)}$, $b_n = \frac{1}{n}$, then $\frac{a_n}{b_n} \to 1$.

and $\sum b_n$ diverges. hence $\sum a_n$ diverges.

(4) $\sum_{n=10}^{\infty} \frac{1}{n \ln n}$

~~Converges. by in~~ diverges. by integral test

$$\int \frac{1}{x} \frac{1}{\ln x} dx = \int \frac{1}{\ln x} d(\ln x) = \ln(\ln x)$$

which goes to ∞ if $x \to \infty$.

(5) $\sum_{n=1}^{\infty} \frac{1}{10^{\ln n}}$

$$10^{\ln n} = e^{(\ln 10)\cdot(\ln n)} = n^{\ln 10}$$

since $\ln 10 > 1$, hence

$\sum \frac{1}{n^{\ln 10}}$ converges.

3. (5 points each) Write down the first few terms of the Taylor expansion for the following functions. More precisely, for Taylor expansion around $x = a$, write down a_0, a_1, a_2 in the expansion $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$.

(1) Taylor expand $(1+x)^{-1}$ around $x = 1/2$

Let $x = \frac{1}{2} + u$, then

$$\frac{1}{1+x} = \frac{1}{1+(\frac{1}{2}+u)} = \frac{1}{\frac{3}{2}+u} = \frac{2}{3}\frac{1}{1+\frac{2}{3}u} = \frac{2}{3}\left(1-\left(\frac{2u}{3}\right)+\left(\frac{2u}{3}\right)^2+\cdots\right)$$

$$= \frac{2}{3} - \left(\frac{2}{3}\right)^2 \left(x-\frac{1}{2}\right) + \left(\frac{2}{3}\right)^3 \left(x-\frac{1}{2}\right)^2 + \cdots$$

(2) Taylor expand $\int_0^x \sqrt{(1+t)(1+2t)}dt$ around $x = 0$

We Taylor expand $\sqrt{(1+t)(1+2t)}$ around $t=0$, to get

$$(1+t)^{\frac{1}{2}} = (1+\tfrac{1}{2}t+\cdots)$$

$$(1+2t)^{\frac{1}{2}} = (1+\tfrac{1}{2}\cdot 2t+\cdots)$$

$$(1+t)^{\frac{1}{2}}(1+2t)^{\frac{1}{2}} = 1+\tfrac{3}{2}t+\cdots$$

$$\int_0^x \sqrt{(1+t)(1+2t)}\,dt = x + \frac{3}{2}\cdot\frac{x^2}{2}+\cdots$$

(3) Taylor expand $\exp(\sqrt{1+x}-1)$ around $x = 0$

$$e^{\sqrt{1+x}-1} = e^{\frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2}\cdot x^2 + \cdots}$$

$$= 1 + \left(\tfrac{1}{2}x - \tfrac{1}{8}x^2+\cdots\right) + \tfrac{1}{2}\left(\tfrac{1}{2}x - \tfrac{1}{8}x^2+\cdots\right)^2 + \cdots$$

$$= 1 + \tfrac{1}{2}x + \left(\tfrac{1}{8}-\tfrac{1}{8}\right)x^2 + \cdots = 1+\tfrac{1}{2}x + 0\,x^2+\cdots$$

(4) Taylor expand $(1-\cos x)/\sin x$ around $x = 0$.

$$\frac{1-\cos x}{\sin x} = \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)}{x-\frac{x^3}{3!}+\cdots}$$

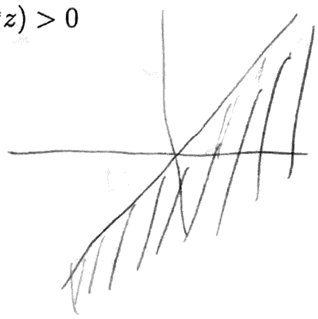
$$= \frac{\frac{1}{2}x - \frac{1}{4!}x^3}{1-\frac{x^2}{3!}+\cdots} = \frac{1}{2}x + 0\cdot x^2 + \cdots$$

4. (10 points, 2 points each) Complex numbers.

(1) If $z = 2019 + 1008i$, then $\mathrm{Im}(z^3 + \bar{z}^3) = ?$ 0

since z^3 and $\bar{z}^3$ are complex conjugate of each other.

(2) Draw the region where $\mathrm{Re}(e^{i\pi/4}z) > 0$



(3) $(1+i)^{12} = ?$

$$1+i = \sqrt{2} \cdot e^{i\pi/4}$$

$$(1+i)^{12} = 2^6 \cdot e^{i\frac{\pi}{4} \cdot 12} = 2^6 e^{i3\pi} = -2^6$$

(4) If $z = 2019 + 1008i$, then $|z/\bar{z}| = ?$. 1

since $|z| = |\bar{z}|$.

(5) How many solution does $z^9 = 10$ have? What are the sum of these solutions?

9 solutions.

they are $10^{\frac{1}{9}}, 10^{\frac{1}{9}} \cdot e^{\frac{2\pi i}{9}}, \cdots, 10^{\frac{1}{9}} \cdot e^{\frac{2\pi i}{9} \cdot 8}$

since $1 + e^{\frac{2\pi i}{9}} + \cdots + e^{\frac{2\pi i}{9} \cdot 8} = 0$.

the sum of the 9 sol'n is 0.

5. (10 points, 5 points each) Complex analytic functions.

(1) Is the following complex valued function analytic?

$$f(x, y) = \cos x \cos y - i \sin x \sin y$$

method 1 let $u = \cos x \cos y$

$v = -\sin x \sin y$.

then $\frac{\partial u}{\partial x} = -\sin x \cos y$, $\frac{\partial u}{\partial y} = -\cos x \sin y$

$\frac{\partial v}{\partial x} = -\cos x \sin y$ $\frac{\partial v}{\partial y} = -\sin x \cos y$.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$. but $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$.

so it is not analytic.

(2) Find the singularities of the following functions, and classify them as branching points, poles or essential singularity

$$\sin(\frac{1}{z}), \quad \sqrt{z(z+2)}, \quad \frac{z}{z+3}, \quad \exp(\frac{z+1}{z^2}), \quad \frac{1}{\sin z}$$

$\sin(\frac{1}{z})$: singular at $z=0$, essential singularity

$\sqrt{z(z+2)}$. : singular at $z=0,-2$. branching points.

$\frac{z}{z+3}$: pole at $z=-3$.

$e^{\frac{z+1}{z^2}}$ essential singularity at $z=0$.

$\frac{1}{\sin z}$: ~~st~~ poles at $z = n\pi$. $n \in \mathbb{Z}$

6. (10 points) Write down the Laurent expansion for

$$\frac{2}{(z+1)(z+3)}$$

in the annulus $1 < |z| < 3$. (Please write down the formula for the coefficients, not just the first few terms.)

$$\frac{2}{(z+1)(z+3)} = \frac{1}{z+1} - \frac{1}{z+3}.$$

for $1 < |z|$, we have

$$\frac{1}{z+1} = \frac{1}{z}\frac{1}{1+\frac{1}{z}} = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} + \cdots + (-1)^n \frac{1}{z^n} + \cdots\right)$$

for $|z| < 3$, we have

$$\frac{1}{z+3} = \frac{1}{3}\frac{1}{1+z/3} = \frac{1}{3}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \cdots + (-1)^n \left(\frac{z}{3}\right)^n + \cdots\right)$$

Hence for $1 < |z| < 3$.

$$\frac{1}{z+1} - \frac{1}{z+3} = -\frac{1}{3} + \frac{z}{3^2} + \cdots + \frac{1}{3}\cdot(-1)^{n+1}\left(\frac{z}{3}\right)^n + \cdots$$

$$+ \frac{1}{z} - \frac{1}{z^2} + \cdots + (-1)^{n+1}\frac{1}{z^n} + \cdots$$

7. (30 points, 5 points each) Evaluate the following contour integrals.

(1)

$$\oint_{|z|=1} \frac{z^3 + 100z^2 + 5z + 7}{z^2} dz = ?$$

$2\pi i \cdot 5$

(2)

$$\oint_{|z|=1} \frac{(z+2)(z+3)}{z(z+4)(z+5)} dz = ?$$

$2\pi i \cdot \frac{2 \cdot 3}{4 \cdot 5}$

(3)

$$\oint_{|z|=2} \frac{e^z}{z(z+1)} dz = ?$$

$2\pi i \cdot \left(1 + \frac{e^{-1}}{-1} \right)$

(4)

$$\oint_{|z|=1} e^{1/z} dz = ?$$

$$\oint_{|z|=1} \left(1 + \frac{1}{z} + \frac{(1/z)^2}{2!} + \cdots \right) dz = 2\pi i \cdot 1$$

(5)

$$\oint_{|z|=2} \frac{1}{z^8+1} dz = ?$$

$$-\oint_{|w|=\frac{1}{2}} \frac{1}{(\frac{1}{w})^8+1} \left(-\frac{dw}{w^2}\right) = \oint_{|w|=\frac{1}{2}} \frac{w^6}{1+w^8} dw = 0$$

(6)

$$\oint_{|z|=1} \frac{1}{\sin(1/z)} dz = ?$$

$$-\oint_{|w|=1} \frac{1}{\sin w} \left(-\frac{dw}{w^2}\right) = \oint_{|w|=1} \frac{1}{w^2 \sin w} dw$$

there is only a pole ~~has~~ of $\frac{1}{w^2 \sin w}$ at $w=0$ inside $|w|=1$.

$$\operatorname{Res}_0 \frac{1}{w^2 \sin w} = \operatorname{Res}_0 \frac{1}{w^2 (w - \frac{w^3}{3!} + \cdots)} = \operatorname{Res}_0 \frac{1}{w^3 (1 - \frac{w^2}{3!} + \cdots)}$$

$$= \operatorname{Res}_0 \frac{1 + (\frac{w^2}{3!} + \cdots) + (\frac{w^2}{3!} + \cdots)^2 + \cdots}{w^3} = 1/6.$$

Hence, result $= 2\pi i \cdot \frac{1}{6}$