Math 121A First Midterm October 8, 2019

Name: ____

- You have 80 minutes to complete the exam.
- This is a closed-book exam. No notes, books, calculators, computers, or electronic aids are allowed.
- All work must be done on this exam packet. If you need more space for any problem, feel free to continue your work on the back of the page. Draw an arrow or write a note indicating this so that the reader knows where to look for the rest of your work.
- Please write neatly. Answers which are illegible for the reader cannot be given credit.

Good Luck!

Question	Points	Score
1	10	
2	10	
3	20	
4	10	
5	10	
6	10	
7	30	
Total	100	

Useful formula for Taylor expansion:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \cdots$$

$$\sin x = x - x^3/3! + x^5/5! + \cdots$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2 \cdot 1}x^2 + \cdots + \binom{p}{n}x^n + \cdots$$

$$\ln(1+x) = x - x^2/2 + x^3/3 + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$

1. (2 points each) Determine if the following sequences converge. Write a short justification.

   (1) $\sin(n\pi/12)$, $n \to \infty$.

   (2) $\frac{1}{n}\sin(n\pi/12)$, $n \to \infty$.

   (3) $\sqrt{n+1} - \sqrt{n}$, $n \to \infty$

   (4) $n^{\ln n}/e^n$, $n \to \infty$

   (5) $\frac{(n+\sin n)^2}{(n+\cos n)^2}$, $n \to \infty$

2. (2 points each) Determine if the following series converge. Write a short justification.

(1) $\sum_{n=1}^{\infty} e^{-n} n^2$

(2) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

(3) $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{(n+3)(n+4)(n+5)}$

(4) $\sum_{n=10}^{\infty} \frac{1}{n \ln n}$

(5) $\sum_{n=1}^{\infty} \frac{1}{10^{\ln n}}$

3. (5 points each) Write down the first few terms of the Taylor expansion for the following functions. More precisely, for Taylor expansion around $x = a$, write down a_0, a_1, a_2 in the expansion $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$.

(1) Taylor expand $(1+x)^{-1}$ around $x = 1/2$

(2) Taylor expand $\int_0^x \sqrt{(1+t)(1+2t)}dt$ around $x = 0$

(3) Taylor expand $\exp(\sqrt{1+x} - 1)$ around $x = 0$

(4) Taylor expand $(1 - \cos x)/\sin x$ around $x = 0$.

4. (10 points, 2 points each) Complex numbers.

(1) If $z = 2019 + 1008i$, then $\mathrm{Im}(z^3 + \bar{z}^3) =?$

(2) Draw the region where $\mathrm{Re}(e^{i\pi/4}z) > 0$

(3) $(1 + i)^{12} =?$

(4) If $z = 2019 + 1008i$, then $|z/\bar{z}| =?$.

(5) How many solution does $z^9 = 10$ have? What are the sum of these solutions?

5. (10 points, 5 points each) Complex analytic functions.

(1) Is the following complex valued function analytic?

$$f(x, y) = \cos x \cos y - i \sin x \sin y$$

(2) Find the singularities of the following functions, and classify them as branching points, poles or essential singularity

$$\sin(\frac{1}{z}), \quad \sqrt{z(z+2)}, \quad \frac{z}{z+3}, \quad \exp(\frac{z+1}{z^2}), \quad \frac{1}{\sin z}$$

6. (10 points) Write down the Laurent expansion for

$$\frac{2}{(z+1)(z+3)}$$

in the annulus $1 < |z| < 3$. (Please write down the formula for the coefficients, not just the first few terms.)

7. (30 points, 5 points each) Evaluate the following contour integrals.

(1)

$$\oint_{|z|=1} \frac{z^3 + 100z^2 + 5z + 7}{z^2} dz = ?$$

(2)

$$\oint_{|z|=1} \frac{(z+2)(z+3)}{z(z+4)(z+5)} dz = ?$$

(3)

$$\oint_{|z|=2} \frac{e^z}{z(z+1)} dz = ?$$

(4)

$$\oint_{|z|=1} e^{1/z} dz = ?$$

(5)

$$\oint_{|z|=2} \frac{1}{z^8+1} dz = ?$$

(6)

$$\oint_{|z|=1} \frac{1}{\sin(1/z)} dz = ?$$