Mathematics 53. Fall Semester 2018 **Professor: Daniel Tataru**

Midterm 2 solutions

(20) 1. In the region $\{y > |x|\}$ introduce hyperbolic coordinates

$$y = e^u \cosh v, \qquad x = e^u \sinh v$$

where the hyperolic sine and cosine are defined by

$$\cosh v = \frac{e^v + e^{-v}}{2}, \qquad \sinh v = \frac{e^v - e^{-v}}{2}$$

a) Compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$. [Hint: $\cosh^2 v - \sinh^2 v = 1$.]

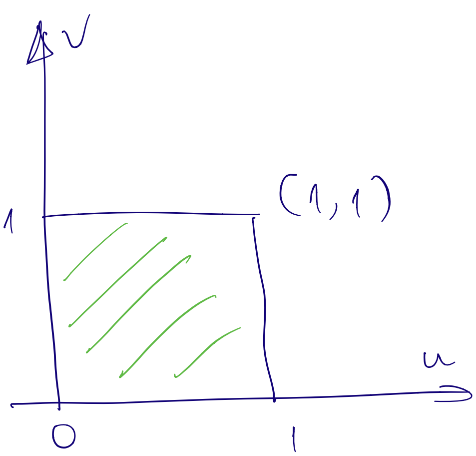
Solution: We have $(\cosh v)' = \sinh v$ and $(\sinh v)' = \cosh v$. Hence th Jacobian is

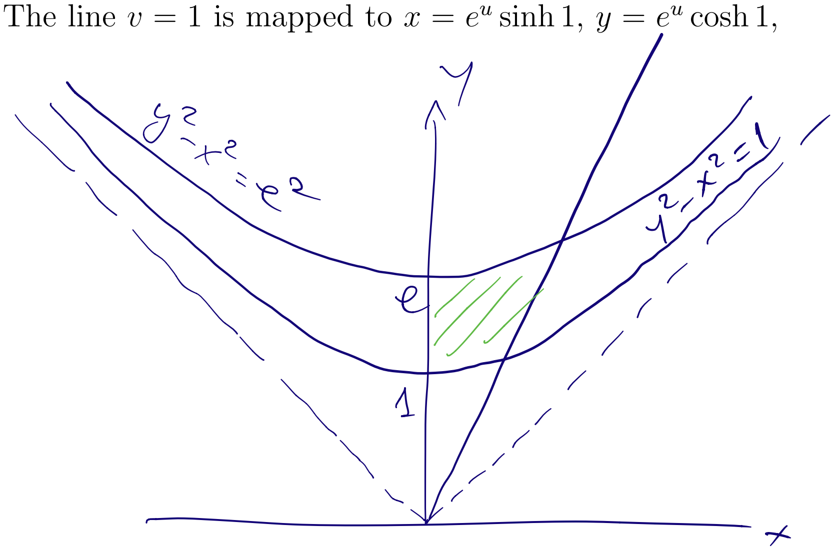
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} e^u \sinh v & e^u \cosh v \\ e^u \cosh v & e^u \sinh v \end{vmatrix} = -e^{2u}$$

b) Sketch the image of the unit square $\{0 \le u, v \le 1\}$ via this change of coordinates.

Solution: The line $u = 0$ is mapped to $x = \sinh v$ and $y = \sinh v$. Using the hint, this gives the hyperbola $y^2 - x^2 = 1$. Similarly, the line $u = 1$ is mapped to the hyperbola $y^2 - x^2 = e^2$.

The line $v = 0$ is mapped to $x = 0$. The line $v = 1$ is mapped to $x = e^u \sinh 1$, $y = e^u \cosh 1$, which gives $x = y \tanh 1$.





(10) 2. Consider the curve $C = \{\sqrt{x} + \sqrt{y} = 1\}$ starting at $(0, 1)$ and ending at $(1, 0)$. Evaluate

$$\int_C y\ dx - x\ dy$$

Solution: We parametrize the curve as

$$x = t^2, \qquad y = (1-t)^2, \qquad t \in [0, 1]$$

Then our integral becomes

$$\int_0^1 (1-t)^2(2t) - t^2(-2(1-t))\ dt = \int_0^1 -2t^2 + 2t\ dt = -\frac{2}{3} + 1 = \frac{1}{3}$$

(30) 3. a) State Green's theorem in the region: [Sketch it first]

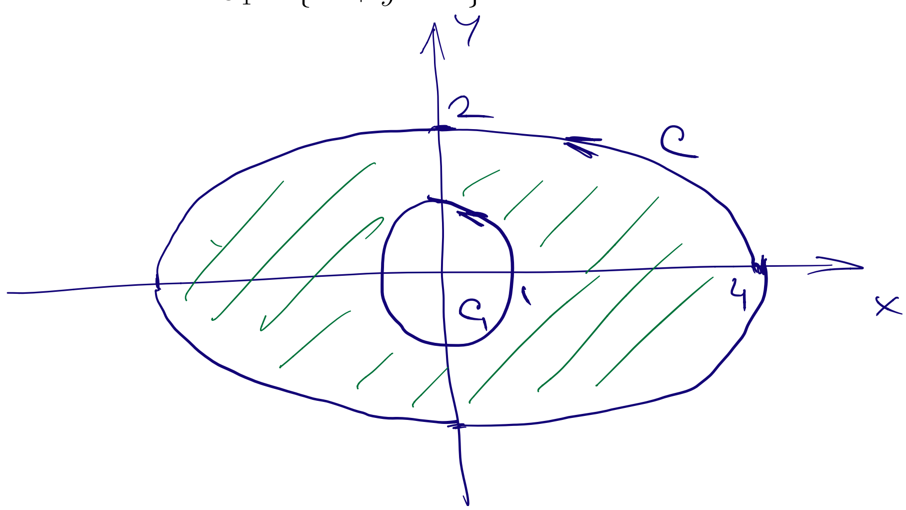
$$D = \{1 \le x^2 + y^2, \quad x^2 + 4y^2 \le 16\}$$

Solution: The boundary of D has two components, an outer one which is the elipse

$$C = \{x^2 + 4y^2 = 16\}$$

and an inner one which is the circle

$$C_1 = \{x^2 + y^2 = 1\}$$



Then Green's theorem for D has the form

$$\int_D Q_y - P_x \, dA = \oint_C Pdx + Qdy - \oint_{C_1} Pdx + Qdy$$

b) Consider the vector field

$$F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

Is it conservative in $\mathbb{R}^2 \setminus \{0\}$? But in the first quadrant $\{x > 0, \ y > 0\}$?

Solution: With $F = (P, Q)$ then $P_y = Q_x$. The first quadrant is simply connected, so F is conservative there. On the other hand $\mathbb{R}^2 \setminus \{0\}$ is not simply connected. Hence we need to check the integral on a contour around the hole. We choose the contour C_1 above. We parametrize C_1 using polar coordinates, $x = \cos\theta, \ y = \sin\theta$. We have

$$\oint_{C_1} Fdr = \int_0^{2\pi} \sin^2\theta + \cos^2\theta \, d\theta = 2\pi.$$

This is nonzero, so F is not conservative in $\mathbb{R}^2 \setminus \{0\}$.

c) For the above F evaluate the integral

$$\oint_C F \cdot dr, \qquad C = \{x^2 + 4y^2 = 16\}.$$

Solution: We use Green's theorem as in part (a) for F as in part (b). This gives

$$\int_C Fdr = \int_{C_1} Fdr = 2\pi.$$

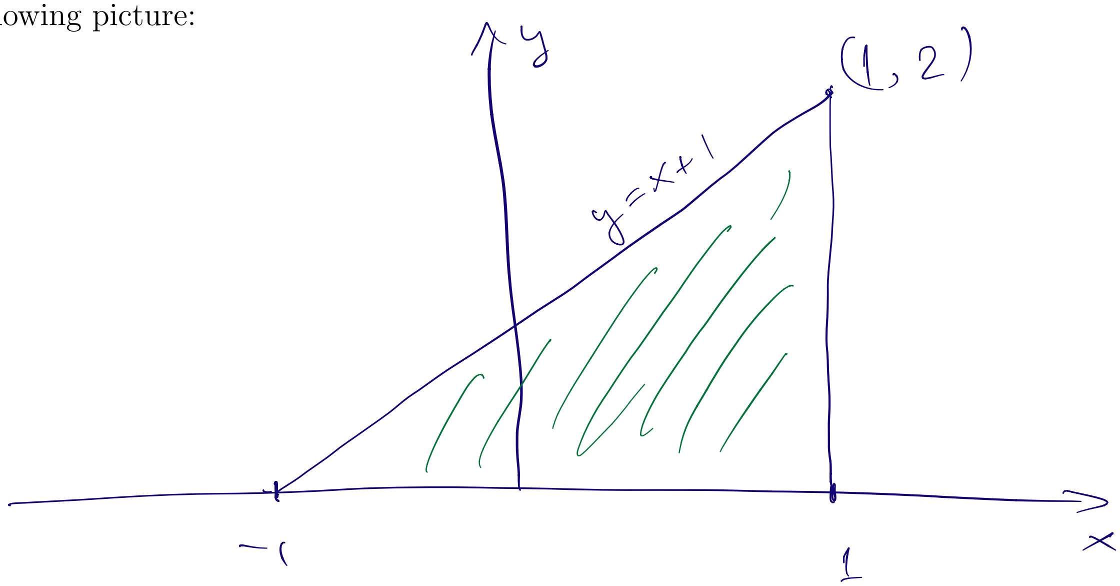
(15) 4. Evaluate the integral

$$I = \int_0^2 \int_{y-1}^1 \sqrt{x^2 + 2x + 2}\, dxdy$$

Solution: The integration domain is described as

$$D = \{0 \le y \le 2,\ y - 1 \le x \le 1\} = \{-1 \le x \le 1,\ 0 \le y \le x + 1\}$$

as in the following picture:



Hence, changing the order of integration we obtain

$$\begin{aligned} I &= \int_{-1}^1 \int_0^{x+1} \sqrt{x^2 + 2x + 2}\, dydx \\ &= \int_{-1}^1 (x+1)\sqrt{(x+1)^2 + 1}\, dx \\ &= \int_0^2 u\sqrt{u^2+1}\, du = \frac{1}{3}(u^2+1)^{\frac{3}{2}}|_0^2 \\ &= \frac{1}{3}(5^{\frac{3}{2}} - 1). \end{aligned}$$

(15) 5. Let D be the square with vertices $(0,1)$, $(1,0)$, $(0,-1)$ and $(-1,0)$. Evaluate

$$I \int_D e^{x+y}(x-y)^{2018} dA$$

Solution: We change variables to

$$u = x + y, \qquad v = x - y$$

or in reverse order

$$x = \frac{u+v}{2}, \qquad y = \frac{u-v}{2}.$$

The Jacobian is

$$J = \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}.$$

Now we need to find out the image of the domain D in the $u-v$ coordinates. The sides of the square D have the equations

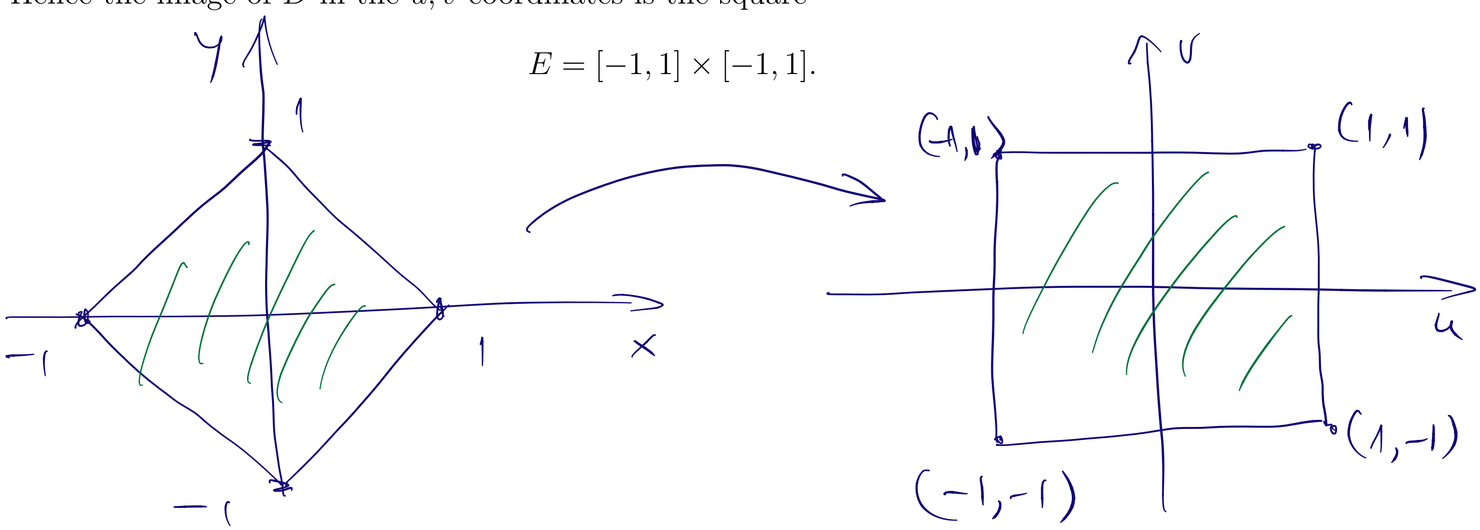
$$x + y = \pm 1, \qquad x - y = \pm 1.$$

Expressed un terms of u, v these become

$$u = \pm 1, \qquad v = \pm 1.$$

Hence the image of D in the u, v coordinates is the square

$$E = [-1,1] \times [-1,1].$$



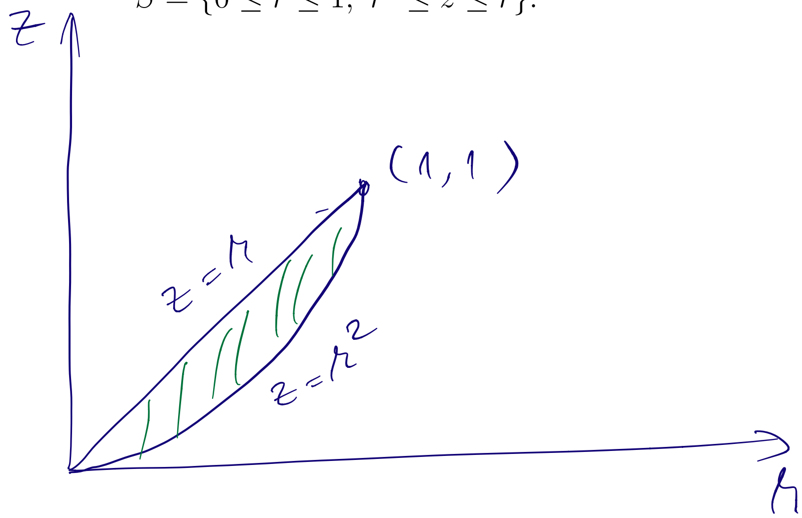
In the new coordinates, the integral takes the form

$$I = \int_E \frac{1}{2} e^u v^{2018}\, dA = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} e^u v^{2018}\, dudv = \frac{1}{2} \cdot e^u|_{-1}^{1} \cdot \frac{1}{2019} v^{2019}|_{-1}^{1} = \frac{e - e^{-1}}{2019}.$$

(10) 6. Consider the solid bounded by the paraboloid $z = x^2 + y^2$ and the cone $z^2 = x^2 + y^2$, with density $\rho(x, y, z) = 6z$. Find its mass.

Solution: In polar coordinates this solid can be described (see picture) as

$$S = \{0 \le r \le 1,\ r^2 \le z \le r\}.$$



The mass is

$$M = \int_S \rho\ dv$$

which in polar coordinates gives

$$M = \int_0^{2\pi} \int_0^1 \int_r^{r^2} 6z\ rdzdrd\theta = \int_0^{2\pi} \int_0^1 3(r^2 - r^4) rdrd\theta = 3 \cdot 2\pi \cdot (\frac{1}{4} - \frac{1}{6}) = \frac{\pi}{2}$$