Mathematics 53. Fall Semester 2018 **Professor: Daniel Tataru**

Midterm 1 solutions

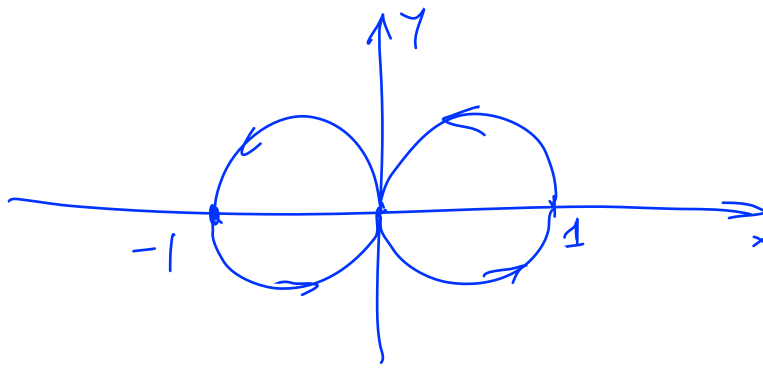
(20) 1. Consider the parametric curve in polar coordinates

$$r = \cos^2\theta, \qquad \theta \in [0, 2\pi]$$

a) Sketch the curve.
b) Compute the area enclosed by the curve.
c) Find the slope of the curve at $\theta = \frac{\pi}{2}$.

Solution:

a) Here it is advisable to either plot several points, or to graph first the function $r = \cos^2\theta$. Our curve is as follows:



b) The area is given by

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} r^2 d\theta = \frac{1}{2}\int_0^{2\pi} \cos^4\theta d\theta \\ &= \frac{1}{8}\int_0^{2\pi} (1+\cos(2\theta))^2 d\theta = \frac{1}{8}\int_0^{2\pi} 1 + 2\cos(2\theta) + \cos^2(2\theta) d\theta \\ &= \frac{1}{8}\int_0^{2\pi} 1 + 2\cos(2\theta) + \frac{1}{2}(1+\cos(4\theta)) d\theta \\ &= \frac{1}{8}\frac{3}{2}\theta + \sin(2\theta) + \frac{1}{8}\sin(4\theta)|_0^{2\pi} = \frac{3\pi}{8} \end{aligned}$$

c) The slope at any given point is evaluated using a consequence of the chain rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dy}{d\theta}}$$

We have $x = \cos^3\theta$ and $y = \sin\theta\cos^2\theta$, so the slope is

$$a(\theta) = \frac{-2\sin^2\theta\cos\theta + \cos^3\theta}{-3\sin\theta\cos^2\theta} = \frac{2\sin^2\theta - \cos^2\theta}{3\sin\theta\cos\theta}$$

At $\theta = \frac{\pi}{2}$ this is indetermined. However, we can compute the limit on the left

$$\lim_{theta\nearrow\frac{\pi}{2}} a(\theta) = +\infty$$

and on the right

$$\lim_{theta\searrow\frac{\pi}{2}} a(\theta) = -\infty$$

and in both cases we see that the slope is infinite, in other words the tangent line to our curve is vertical.

(20) 2. Consider the points $P = (0, 1, 1)$ and $Q = (1, 0, 1)$, and let u, v be their position vectors. Calculate/describe:

a) The triple product $u \cdot (v \times u)$.

b) The area of the parallelogram with sides u, v.

c) The parametric line L through P in the direction v.

d) The distance between the point Q and the line L.

Solution:

a) The triple product is zero since two of the vectors coincide.

b) The area is given by $A = |u \times v|$. We compute the cross product

$$u \times v = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = i + j - k$$

so $A = \sqrt{3}$.

c) The parametric line is given by:

$$x(t) = t, \qquad y(t) = 1, \qquad z(t) = 1 + t.$$

d) The distance d between a point on L and Q is given by

$$d^2 = (t-1)^2 + 1 + t^2 = 2t^2 - 2t + 2 = 2(t - \frac{1}{2})^2 + \frac{3}{2}$$

This is minimized at $t = \frac{1}{2}$, and the minimum is

$$d_{min} = \sqrt{3/2}.$$

(20) 3. Consider the parametric curve $\mathbf{r}(t) = (2t, \log t, t^2)$ for $t \in [1,4]$.
a) Find its length.
b) Find its curvature at $t = 1$.
c) Find its unit tangent and normal vector, also at $t = 1$.

Solution: a) We have

$$\mathbf{r}'(t) = (2, \frac{1}{t}, 2t), \qquad |\mathbf{r}'(t)| = \sqrt{\frac{1}{t^2} + 4 + 4t^2} = \frac{1}{t} + 2t$$

(here we note that $1 + 2t > 0$ for $t \in [1,4]$). Then the length is

$$L = \int_1^4 |\mathbf{r}'(t)| dt = \int_1^4 \frac{1}{t} + 2t dt = \ln t + t^2|_1^4 = 15 + \ln 4$$

b) The curvature is

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$$

We have

$$\mathbf{r}'' = (0, -\frac{1}{t^2}, 2)$$

and

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} i & j & k \\ 2 & \frac{1}{t} & 2t \\ 0 & -\frac{1}{t^2} & 2 \end{vmatrix} = \frac{4}{t} i - 4j - \frac{2}{t^2} k, \qquad |\mathbf{r}' \times \mathbf{r}''| = \sqrt{\frac{4}{t^4} + \frac{16}{t^2} + 16} = \frac{2}{t^2} + 4$$

Hence the curvature at $t = 1$ is

$$\kappa = \frac{6}{27} = \frac{2}{9}.$$

c) The unit tangent vector is

$$T(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \left(\frac{2t}{1+2t^2}, \frac{1}{1+2t^2}, \frac{2t^2}{1+2t^2} \right)$$

The unit normal vector N is given by

$$N = \frac{T'}{|T'|}$$

We compute

$$T'(t) = \left(\frac{2-4t^2}{(1+2t^2)^2}, \frac{-4t}{(1+2t^2)^2}, \frac{4t}{(1+2t^2)^2} \right)$$

At $t = 1$ we have

$$T = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right), \qquad T' = \left(\frac{-2}{9}, \frac{-4}{9}, \frac{4}{9} \right), \qquad N = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right).$$

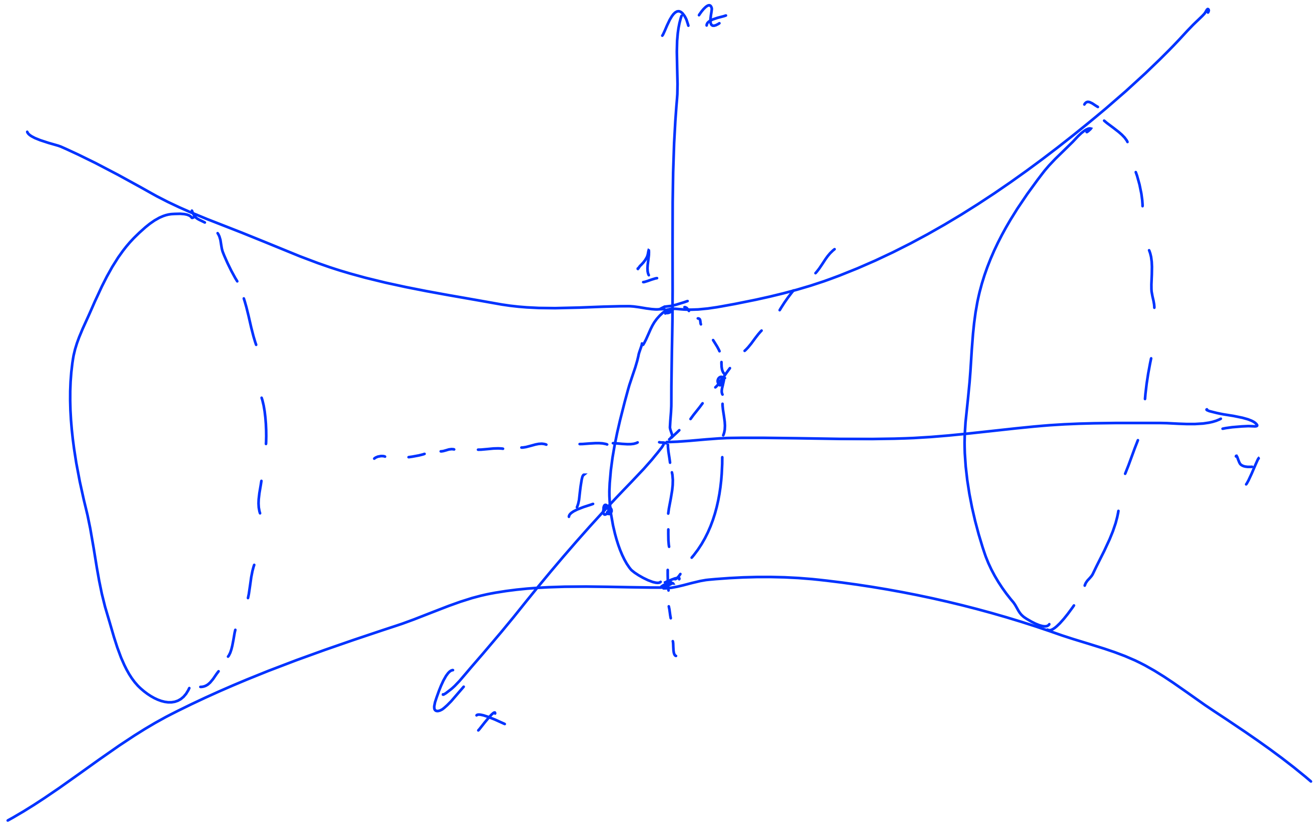
To double check, one may verify that T and N are orthogonal.

(20) 4. Let S be the surface

$$x^2 - y^2 + z^2 = 1$$

a) Identify it and sketch it.
b) Find the equation of its tangent plane at the point $(1, 2, 2)$.

Solution: a) The surface is a hyperboloid with one sheet, which intersects the $x - z$ plane on the unit circle but does not intersect the y axis:



b) S is a level set of the function $f(x, y, z) = x^2 - y^2 + z^2$ with gradient

$$\nabla f = (2x, -2y, 2z).$$

The gradient is perpendicular to the tangent plane, so at $(1, 2, 2)$ the normal vector is $(2, -4, 4)$. Hence, the equation of the tangent plane is

$$2(x - 1) - 4(y - 2) + 4(z - 2) = 0$$

or equivalently

$$x - 2y + 2z = 1.$$

(20) 5. Consider the function $f(x, y) = 2x^3 + y^2 - 6xy + 4y$.

a) Find its local maximum and minimum values and saddle points.

b) Find its global maximum and minimum inside the triangle with vertices $(0, 0)$, $(0, 6)$ and $(6, 0)$.

Solution:

a) We have

$$f_x = 6x^2 - 6y, \qquad f_y = 2y - 6x + 4.$$

To find the critical points we set both to zero and solve for (x, y). From the second equation $y = 3x - 2$, and substituting in the first we get

$$x^2 - 3x + 2 = 0$$

which has two solutions $x = 1, 2$. Then the critical points are

$$P = (1, 1), \qquad Q = (2, 4).$$

To classify them we use the second derivative test. We have

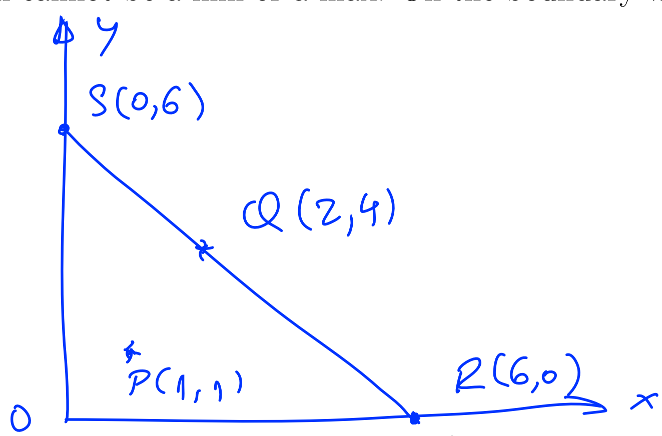
$$f_{xx} = 12x, \qquad f_{yy} = 2, \qquad f_{xy} = -6$$

and

$$D = f_{xx} + f_{yy} - f_{xy}^2 = 24x - 36$$

Verifying the signs of f_{xx}, f_{yy} and D at P and Q we see that P is a saddle point and Q is a local minimum point.

b) Here we need to check what happens inside the triangle and on its boundary. Inside we only have the point P, which cannot be a min or a max. On the boundary we consider the three sides: .



i) On OR we have $y = 0$ and $x \in [0, 6]$. Then $f(x, y) = 2x^3$, which has no critical points inside $(0, 6)$.

ii) On OS we have $x = 0$ and $y \in [0, 6]$. Then $f(x, y) = y^2 + 4y$, which has no critical points inside $(0, 6)$.

iii) On RS we have $y = 6 - x$ and $x \in [0, 6]$. Then

$$f(x, y) = g(x) = 2x^3 + (x - 6)^2 - 6x(6 - x) + 4(6 - x) = 2x^3 + 7x^2 - 52x + 24$$

We compute

$$g'(x) = 6x^2 + 14x - 52 = (x - 2)(6x + 26)$$

(here we knew that 2 must be a root since $Q \in RS$) which has the critical point $x = 2$ inside $(0, 6)$. This corresponds to $y = 4$, so we have recovered the point Q. [One could also use Lagange multipliers for this last part.]

To summarize, our candidates for min/max remain the points O, R, S, Q. We evaluate f,

$$f(O) = 0, \qquad f(R) = 432, \qquad f(S) = 88, \qquad f(Q) = 0.$$

Hence the minimum of f over the triangle is 0 and the maximum is 432.