Final Exam Solutions

(25) 1. a) (5) For a function $f(x,y)$ in $\mathbb{R}^2$ define its Laplacian Δf (also denoted in the book by $\nabla^2 f$).

Solution: We have

$$\Delta f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial_x^2} + \frac{\partial^2 f}{\partial_x^2}$$

b) (10) Let g be a twice differentiable function of one variable and $f(x,y) = g(r)$ where r is the radius in polar coordinates. Compute the expression Δf in terms of g and its derivatives.

Solution: We first use chain rule to compute

$$\frac{\partial f}{\partial_x} = g'(r)\frac{\partial r}{\partial x} = g'(r)\frac{x}{\sqrt{x^2+y^2}}$$

and then differentiate again

$$\frac{\partial^2 f}{\partial_x^2} = g''(r)\frac{x^2}{x^2+y^2} + g'(r)\frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}$$

By symmetry,

$$\frac{\partial^2 f}{\partial_y^2} = g''(r)\frac{y^2}{x^2+y^2} + g'(r)\frac{1}{\sqrt{x^2+y^2}} - \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

Summing up the last two relations,

$$\Delta f = g''(r) + \frac{1}{r}g'(r)$$

c) (10) Check whether the function $f(x,y) = \ln(x^2+y^2)$ solves the Laplace equation $\Delta f = 0$.

Solution:
We apply part (b) with $g(r) = \ln(r^2) = 2\ln r$. Then

$$g'(r) = \frac{2}{r}, \qquad g''(r) = -\frac{2}{r^2}$$

therefore

$$\Delta \ln(x^2+y^2) = \frac{2}{r^2} - \frac{2}{r^2} = 0.$$

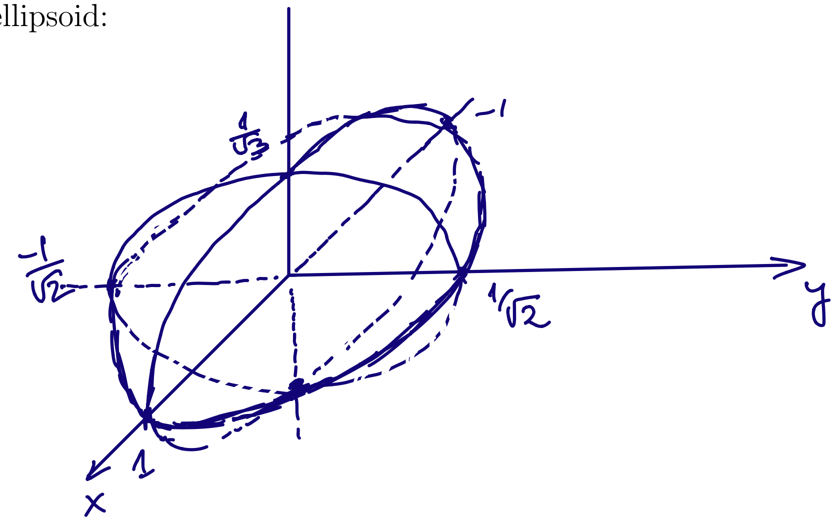
Hence f solves the Laplace equation.

(25) 2. An asteroid A is described by the equation

$$x^2 + 2y^2 + 3z^2 \leq 1.$$

a) (5) What is the shape of the asteroid ? Sketch it !

Solution: This is an ellipsoid:



b) What is the equation for the tangent plane at some point $Q = (x_0, y_0, z_0)$ on the surface ?
Solution:
Given a point $Q = (x_0, y_0, z_0)$ on the surface, the tangent plane P has the equation

$$P: \qquad xx_0 + 2yy_0 + 3zz_0 = 1$$

c) (10) What are the points on its surface which are visible from a spaceship located at coordinates $S = (1, 1, 1)$?

Solution: A point Q on the surface is visible from the spaceship if the tangent plane at that point separates the asteroid from the spaceship. The above plane P separates $\mathbb{R}^3$ into two half-spaces

$$H_1 = \{xx_0 + 2yy_0 + 3zz_0 \leq 1\}, \qquad H_2 = \{xx_0 + 2yy_0 + 3zz_0 \geq 1\}$$

We have $(0, 0, 0) \in H_1$ so $A \subset H_1$. Hence the point Q is visible from S if $S \in H_2$. This gives the equation

$$x + 2y + 3z \geq 1.$$

Hence the set V of visible points on the surface is

$$V = \{x^2 + 2y^2 + 3z^2 = 1, \quad x + 2y + 3z \geq 1.\}$$

(20) 3. Consider the curve γ which is the portion of the intersection of the hyperboloid $H = \{x^2 + y^2 - z^2 = 1\}$ with the cylinder $C = \{y = x^2\}$ between the points $P = (-1, 1, 1)$ and $Q = (1, 1, 1)$.

a) Find the direction of the tangent vector to γ at $(1, 1, 1)$.

Solution: The tangent line to γ at Q is the intersection of the tangent planes to H and C at Q. These have normals $N_H = (2, 2, -2)$ respectively $N_C = (2, -1, 0)$, which are obtained by computing the gradients of the defining functions above, namely $x^2 + y^2 - z^2$ and $x^2 - y$. Then a tangent vector to γ is given by

$$T = N_H \times N_C = (-2, -4, -6)$$

b) Evaluate the integral

$$I = \int_\gamma F \cdot dr, \qquad F = \left(\frac{1}{1+y+z}, -\frac{x}{(1+y+z)^2}, 2z - \frac{x}{(1+y+z)^2} \right)$$

Solution: The vector field F is easily seen to be conservative,

$$F = \nabla f, \qquad f(x, y, z) = \frac{x}{1+y+z} + z^2$$

Then

$$I = \int_\gamma F \cdot dr = f(Q) - F(P) = \frac{2}{3}$$

Note: There was a subtle typo in part (b), which is that the point P should have been $(1, 1, -1)$. This is because the intersection of the two surfaces consists of two separate curves, and as given the points P and Q are on different curves. If you did not notice that and solved the problem as above you get full credit. If instead you observed this issue and concluded that the problem cannot be solved then you also get full credit. Partial credit will be awarded for progress in either direction.

(20) 4. Find the maximum of the function

$$f(x, y, z) = xyz + x + y + z$$

within the sphere $S = \{x^2 + y^2 + z^2 \leq 1\}$.

Solution: We first look for critical points inside the sphere. This gives the equation $\nabla f = 0$, i.e.

$$1 + yz = 0, \qquad 1 + xz = 0, \qquad 1 + xy = 0$$

or

$$yz = -1, \qquad xz = -1, \qquad xy = -1.$$

This has no solutions, as is easily seen by multiplying the three equations to obtain $x^2y^2z^2 = -1$. Thus there are no critical points inside. It remains to look for critical points on the boundary ∂S. We use Lagrange multipliers, obtaining the equations

$$1 + yz = 2\lambda x, \qquad 1 + xz = 2\lambda y, \qquad 1 + xy = 2\lambda z.$$

Subtracting the first two equations we get

$$z(y - x) = 2\lambda(x - y),$$

therefore

$$x = y \qquad \text{or} \qquad z = -2\lambda$$

We also obtain all circular permutations of this. Examining all possibilities, we must have either $x = y = z$ or $x = y = -2\lambda$ (or some permutation of the latter). In the first case using the equation of the unit sphere $x^2 + y^2 + z^2 = 1$ we obtain either

$$x = y = z = \frac{1}{\sqrt{3}}$$

or

$$x = y = z = -\frac{1}{\sqrt{3}}$$

In the second case, we must have $z = \dfrac{1 + 4\lambda^2}{2\lambda}$ which is not acceptable as it would give $|z| \geq 2$, not possible on the sphere.

So we have two critical points,

$$P = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), \qquad Q = (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$$

Evaluating f at the two points we have

$$f(P) = \frac{10}{3\sqrt{3}}, \qquad f(Q) = -\frac{10}{3\sqrt{3}}.$$

The largest of these is the maximum of f inside S.

(30) 5. Evaluate the following integrals:

a)

$$I = \int_0^1 \int_0^1 \sin(\max\{x^2, y^2\})\, dx\, dy$$

Solution: The argument of sine is x^2 or y^2 depending on which is larger. So we split our integral in two symmetric regions,

$$D_1 = \{0 \le x \le y \le 1\}, \qquad D_2 = \{0 \le x \le y \le 1\},$$

Then we have

$$I = 2\iint_{D_1} \sin(y^2) dV = \int_0^1 \int_0^y 2\sin(y^2)\, dx\, dy = \int_0^1 2y \sin(y^2)\, dy = -\cos(y^2)\Big|_0^1 = 1 - \cos 1.$$

b)

$$J = \int_0^1 \int_0^{\sqrt{1-z^2}} \int_z^{\sqrt{1-y^2}} x^2 y\, dx\, dy\, dz$$

Solution: We examine the integration domain D, which is described by the relations

$$D = \{0 \le z \le 1, \quad 0 \le y \le \sqrt{1-z^2}, \quad z \le x \le \sqrt{1-y^2}\}$$

We rewrite it as

$$D = \{x, y, z \ge 0, \quad y^2 + z^2 \le 1, \quad z \le x, \quad x^2 + y^2 \le 1\}$$

and finally as

$$D = \{x, y \ge 0, \quad 0 \le z \le x, \quad x^2 + y^2 \le 1\}$$

Using cylindrical coordinates in D, we have

$$\begin{aligned}
J &= \iiint_D x^2 y\, dV \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{r\cos\theta} r^3 \cos^2\theta \sin\theta\, r dz\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 r^5 \cos^3\theta \sin\theta\, dr\, d\theta \\
&= \frac{1}{4} \int_0^{\frac{\pi}{2}} \cos^3\theta \sin\theta\ d\theta \cdot \int_0^1 r^5\, dr \\
&= \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{6} \\
&= \frac{1}{96}
\end{aligned}$$

(15) 6. Let Ω be the solid bounded from above by the sphere $x^2+y^2+z^2=2$ and from below by the paraboloid $z=x^2+y^2$. Find its volume.

Solution: We need to compute

$$V=\iiint_D 1\,dV$$

We represent Ω as

$$\Omega=\{x^2+y^2\leq z\leq\sqrt{2-x^2-y^2}\}$$

Then (x,y) must be in the domain

$$D=\{x^2+y^2\leq 1\}$$

Thus we have

$$V=\iint_D(\sqrt{2-x^2-y^2}-(x^2+y^2))dA$$

In polar coordinates this becomes

$$V=\int_0^{2\pi}\int_0^1(\sqrt{2-r^2}-r^2)r\,dr\,d\theta=2\pi(-\frac{1}{3}(2-r^2)^{\frac{3}{2}}-\frac{1}{4}r^4)\Big|_0^1=\frac{2\pi}{3}(2^{\frac{3}{2}}-1)-\frac{\pi}{2}=\frac{8\sqrt{2}-7}{6}$$

(25) 7. a) (10) Carefully state Stokes's Theorem.

b) (15) The curve C is the intersection of the cylinder $x^2+y^2=1$ with the hyperbolic paraboloid

$$z = x^2 - y^2$$

taken with the counterclockwise orientation. Evaluate the line integral

$$I = \oint_C F \cdot d\mathbf{r}, \qquad F = (x-y, x+y, z^7)$$

Solution: We use Stokes' theorem for the surface

$$S = \{z = x^2 - y^2, \quad x^2 + y^2 = 1\}$$

with the orientation induced by the upward normal. Then

$$\oint_C F \cdot d\mathbf{r} = \iint_S \mathrm{curl}\ F \cdot dS$$

We have $\mathrm{curl}F = (0,0,2)$, while, since S is the graph of the function $f(x,y) = x^2 - y^2$ in the domain $D = \{x^2 + y^2 \leq 1\}$,

$$dS = (-f_x, -f_y, 1)\, dxdy$$

Hence we obtain

$$I = \iint_D 2\, dA = 2\pi$$

(20) 8. Consider a pizza given by the planar domain

$$D = \{4x^2 + y^2 \leq 4y\}$$

and with density $\rho(x, y) = 4y - y^2$. Find its mass.

Solution: We need to compute the integral

$$m = \iint_D 4y - y^2 \, dA$$

Completing the squares, we write

$$D = \{4x^2 + (y-2)^2 \leq 4\}$$

which is an ellipse centered at $(0, 2)$. For this it is natoral to introduce coordinates

$$x = r\cos\theta, \qquad y = 2 + 2r\sin\theta, \qquad r \in [0, 1], \quad \theta \in [0, \pi]$$

where the Jacobian is easily computed as

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = 2r$$

Then

$$dA = 2r \, dr \, d\theta$$

Hence changing coordinates we get

$$m = \int_0^{2\pi} \int_0^1 4(1 - r^2 \sin^2\theta) \, 2r \, dr \, d\theta = 8\pi - 2\pi = 6\pi$$

(20) 9. Evaluate the integral

$$I = \iint_S z^{2018}\, dS, \qquad S = \{x^2 + y^2 + z^2 = 1\}$$

Solution 1: [As a surface integral] Using spherical coordinates

$$\mathbf{r} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

we compute

$$\mathbf{r}_\phi = (-\cos\phi\cos\theta, -\cos\phi\cos\theta, \sin\phi), \qquad \mathbf{r}_\theta = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0)$$

and

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = (-\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, -\sin\phi\cos\phi), \qquad |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi$$

Hence

$$I = \int_0^{2\pi}\int_0^{\pi} \cos\phi^{2018}\sin\phi\, d\phi\; d\theta = 2\pi\int_{-1}^{1} u^{2018} du = \frac{4\pi}{2019}$$

where we have substituted $u = -\cos\phi$.

Solution 2: [Using the divergence theorem] The normal vector to the sphere is $n = (x, y, z)$. Hence defining $F = (0, 0, z^{2017})$ we have $F \cdot n = z^{2018}$. Thus

$$I = \int_S F \cdot dS.$$

Since div $F = 2017z^{2016}$, by the divergence theorem we get

$$I = \int_B 2017z^{2016}\, dV, \qquad B = \{x^2 + y^2 + z^2 \leq 1\}$$

In spherical coordinates this becomes

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} 2017(\rho\cos\phi)^{2016}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

and the integration in ρ and ϕ yields the same result as above.