UNIVERSITY OF CALIFORNIA AT BERKELEY

Physics 7C – (Aganagic)

Fall 2019

FIRST MIDTERM – SOLUTIONS

Problem 1 (30 points)

You are given a slightly flexible spherical mirror that can bend to a range of radii of curvature $0.8\text{m} \leq R \leq 10\text{m}$. You stand with your back to a giant sequoia tree which has a height of 80m.

(a) If the tree is 19.5m behind you and you hold the mirror out at a relaxed arm's length ($x = 50$ cm), can you focus the image of the tree on your eye by bending the mirror to a specific curvature? If so, how tall is the image of the tree? If not, how much further would you need to bend the mirror to focus the image?

The mirror equation gives

$$\frac{1}{f} = \frac{1}{0.5\text{m}} + \frac{1}{19.5\text{m} + 0.5\text{m}} = (2 + 0.05)\text{m}^{-1} \quad \Rightarrow \quad f = \frac{20}{41}\text{m}. \tag{1}$$

This corresponds to a radius of curvature $R = \frac{40}{41}\text{m} > 0.8\text{m}$, and so this image can be focused on our eye within the flexibility of the mirror. The magnification is

$$M = -\frac{s_i}{s_o} = -\frac{0.5}{20} = -\frac{1}{40}. \tag{2}$$

So the image is inverted (which the problem doesn't ask for), and 40 times smaller for a height of 2m.

(b) Next you allow the mirror to relax to its maximum radius of curvature and give it to your friend to hold 8m away from you. How close to the tree should you stand so that the image of the tree is again focused on your eye?

At its maximum radius of curvature, the mirror has a focal length $f = 5\text{m}$. The image distance is 8m. The mirror equation for this situation reads

$$\frac{1}{5\text{m}} = \frac{1}{8\text{m}} + \frac{1}{d + 8\text{m}}, \tag{3}$$

which gives $d = (16/3)\text{m}$.

(c) Last, you bring the mirror closer so that it is now 10cm from your face. If you want the image of the tree to be 8cm tall, where should you stand and how far should you bend the mirror? Is this possible, or will the mirror break?

The image distance is now 0.1m. An 8cm image requires a magnification magnitude of 1/1000, which produces a distance

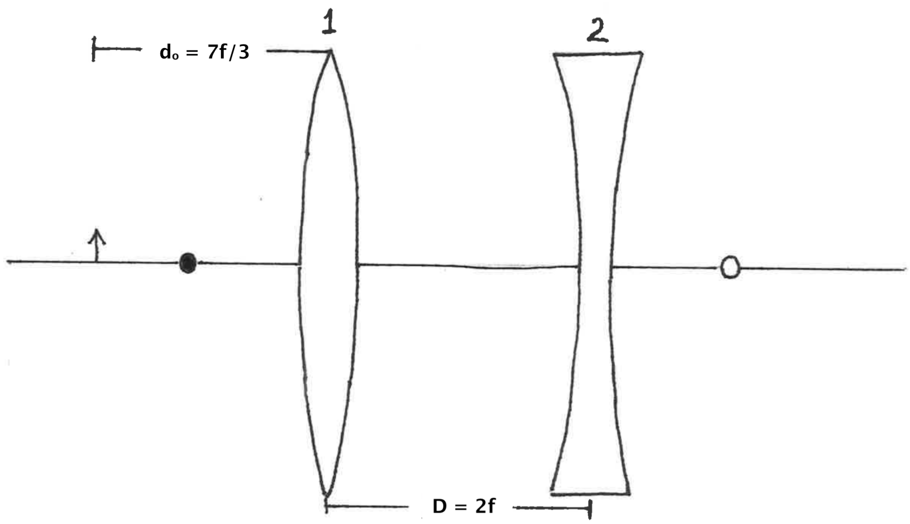
$$|M| = \frac{1}{1000} = \frac{0.1}{d + 0.1} \quad \Rightarrow \quad d = 99.9\text{m}. \tag{4}$$

The mirror equation with these distances gives

$$\frac{1}{f} = \frac{1}{0.1\text{m}} + \frac{1}{100\text{m}} \quad \Rightarrow \quad f = \frac{100}{1001}\text{m} \approx 0.1\text{m}, \tag{5}$$

which is well outside the curvature bounds and will break the mirror.

Problem 2 (30 points)



Two thin lenses, one converging and one diverging, are aligned horizontally. The two foci of the converging lens (number 1) are a distance f from its center; one such focus is shown by the filled circle. Similarly, the two foci of the diverging lens (number 2) are also a distance f from its center; the open circle shows one of these foci. The lens centers are a distance $D = 2f$ apart.

A small, upright object (arrow in the sketch) is placed a distance $d_o^1 = 7f/3$ in front of lens 1. If lens 2 were absent, lens 1 would form an image of this object.

(a) Find d_i^1, the distance of the image from lens 1. Is this image virtual or real?

The lens equation

$$\frac{1}{f} = \frac{3}{7f} + \frac{1}{d_i^1} \tag{6}$$

gives an image distance of $d_i^1 = 7f/4$. Positive image distance means that this is a real image.

With lens 2 in place, the image you found in (a) does not actually form. However, the combined lenses do form an image.

(b) Find d_i^2, the distance of this image from lens 2. Is this image virtual or real?

The image formed by lens 1 will be the object for lens 2. A distance $d_i^1 = 7f/4$ from the right of lens 1, this "object" is a distance $d_o^2 = D - 7f/4 = f/4$ to the left of lens 2. The lens equation for this second lens is

$$-\frac{1}{f} = \frac{4}{f} + \frac{1}{d_o^2}, \tag{7}$$

which gives $d_o^2 = -f/5$. Negative image distance means that this is a virtual image.

(c) Carefully draw a ray diagram showing how the image you found in (b) is formed. You need not show in detail the formation of the *first* image; just place it in your drawing, using the location and orientation you found in (a).

Lines should go in parallel and out through the focal point, or directly through the center of the lens. The final image should be formed between the center of the two lenses and the image from lens 1, and be inverted and virtual.

Problem 3 (40 points)

Consider an electromagnetic wave with electric field

$$\mathbf{E}(y,t) = (ky + \omega t)^4 \hat{\mathbf{x}}, \tag{8}$$

where $k, \omega > 0$.

(a) Show that this satisfies the wave equation. What is the speed of the wave propagation? Suppose this wave travels in a material of index of refraction n_1. How is the speed of the wave propagation related to the speed of light c in vacuum?

Simply compute:

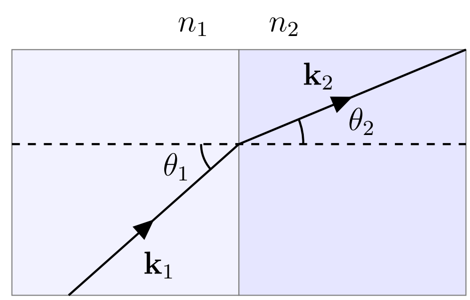
$$\frac{\partial^2 E}{\partial t^2} = 12\omega^2 (ky + \omega t)^2, \qquad \frac{\partial^2 E}{\partial y^2} = 12k^2 (ky + \omega t)^2. \tag{9}$$

This satisfies the wave equation with speed $v^2 = \omega^2/k^2$. In the material with index n_1, the speed is $\nu_1 = c/n_1$.

(b) What is the direction of propagation of this wave? Along what axis does the corresponding magnetic field oscillate?

This wave is moving in the $-y$-direction. Since $\mathbf{S} \propto -\hat{\mathbf{y}}$ and $\mathbf{E} \propto \hat{\mathbf{x}}$, the definition of the Poynting vector gives $-\hat{\mathbf{y}} = \hat{\mathbf{x}} \times \hat{\mathbf{B}}$, or $\hat{\mathbf{B}} = \hat{\mathbf{z}}$.

Now suppose that this wave enters a new material with index of refraction n_2, as in the figure below. Indicate what x, y and z directions from (8) may correspond to in the figure. (This coordinate system is adapted to the wave in the first material, and is not so great for the second. Do not worry about that, it will not be relevant for the rest.)



(c) As the electromagnetic wave transitions from the first material to the second, the frequency of the wave must stay constant. Use this to show that $k_1/k_2 = n_1/n_2$, where k_1 is the wavenumber of the wave in the first material (and similarly for k_2).

Straightforwardly,

$$c = n_1 \nu_1 = n_2 \nu_2 = n_1 \frac{\omega}{k_1} = n_2 \frac{\omega}{k_2} \quad \Rightarrow \quad \frac{n_1}{k_1} = \frac{n_2}{k_2} \quad \Rightarrow \quad \frac{k_1}{k_2} = \frac{n_1}{n_2}. \qquad (10)$$

(d) The wave vector $\vec{k}$ is a vector of magnitude k, in the direction of wave propagation. As any vector, $\vec{k}$ can be decomposed as $\vec{k} = \vec{k}_{||} + \vec{k}_{\perp}$ into a component $\vec{k}_{\perp}$ along the normal to the surface, and component $\vec{k}_{||}$, which is parallel to it. Like the frequency, it turns out that $\vec{k}_{||}$ must be the same on both sides of the surface. Use this fact, and the result from (c) to derive Snell's law. (The condition $\vec{k}_{||}$ is the same on both sides of the surface is the same as asking for wave-fronts, *i.e.* surfaces of constant phase, to match up.)

The parallel components are given by

$$k_{\parallel} = k_1 \sin\theta_1 = k_2 \sin\theta_2. \tag{11}$$

Plugging in the result of (c) gives Snell's law.
