Lecture 2 & 3 Midterm 1 Solutions – Fall 2019

Problem 1

1)

$V_{rms} = u_1 = \sqrt{\frac{3K_BT_1}{m_1}}$ (3 points)

$\boxed{T_1 = \frac{{u_1}^2 * m_1}{3K_B}}$ (2 points)

$V_{rms} = u_2 = \sqrt{\frac{3K_BT_2}{m_2}}$ (3 points)

$\boxed{T_2 = \frac{{u_2}^2 * m_2}{3K_B}}$ (2 points)

2)

Since the system is isolated, $E_{int,\ inital} = E_{int,\ final}$ (2 points)

$E_{int,1} + E_{int,2} = E_{int,\ final}$

$(3/2)n_1RT_1 + (5/2)n_2RT_2 = (1/2)RT(3n_1 + 5n_2)$ (2 points)

$\boxed{T = \frac{3n_1T_1 + 5n_2T_2}{3n_1 + 5n_2}}$ (2 points)

$P = \frac{\mathrm{nRT}}{\mathrm{V}}$ (3 points)

$\boxed{P = \frac{(n_1 + n_2)\mathrm{R}}{V_1 + V_2} * \frac{3n_1T_1 + 5n_2T_2}{3n_1 + 5n_2}}$ (1 points)

Problem 2

2. (a) The total amount of heat transferred to the water will come from two contributions: (1) heating the mass m_w of water to $T_f = 100°\text{C}$ and (2) converting the mass m_s of water to steam. The heat from (1) is calculated using the heat capacity equation $Q_{w,1} = m_w c_w \Delta T = m_w c_w (T_f - T_0)$, and the heat from (2) is calculated using latent heats as $Q_{w,2} = m_s L_v$, where c_w is the heat capacity of water and L_v is the heat of vaporization of water. The total heat transferred by the water is then the sum of these, $Q_w = Q_{w,1} + Q_{w,2} = m_w c_w (T_f - T_0) + m_s L_v$.

   (b) The bowl only absorbs heat in changing its temperature, so we use the heat capacity to calculate that $Q_b = m_b c_b \Delta T = m_b c_u (T_f - T_0)$, where c_u is the heat capacity of copper, the material out of which the bowl is made.

   (c) We now have the total heat absorbed by the bowl/water system, $Q_b + Q_w$. By conservation of energy, this must be equal to the heat lost by the cylinder. This magnitude of this quantity is given by using the heat capacity equation again, $Q_c = m_c c_u (T_h - T_f)$, where T_h is the original temperature of the cylinder and again c_u is the heat capacity of copper. We can then set $Q_b + Q_w$ equal to Q_c to solve for T_h:

$$
\begin{aligned}
Q_c = Q_b + Q_w \Longrightarrow m_c c_u (T_h - T_f) &= m_w c_w (T_f - T_0) + m_s L_v + m_b c_u (T_f - T_0) \\
&= (m_w c_w + m_b c_u)(T_f - T_0) + m_s L_v \\
\Longrightarrow T_h = T_f + \frac{1}{m_c c_u} &\left((m_w c_w + m_b c_u)(T_f - T_0) + m_s L_v\right).
\end{aligned}
$$

Note: Given the information we have, there are two assumptions we can make: (1) not all of the water boils away ($m_s < m_w$) or (2) all of the water is converted to steam ($m_s = m_w$). In the first case, which is assumed in the above solution, the final temperature of the system must be $T_f = 100°\text{C}$. On the one hand, since there is water remaining in the bowl, the temperature must be no greater than 100°C because the temperature of water cannot exceed its boiling point. However, because we know that some of the water is vaporized to steam, the temperature must also be at least 100°C, which leads to the conclusion that $T_f = 100°\text{C}$. In the second case, T_f can be pretty much any temperature that is at least 100°C, in which case we would have to change some things to our solution to account for the fact that the steam can absorb some heat to change its temperature from 100°C to T_f. However, it is safe to assume that case one is the situation to which the question refers, since we are given a different variable m_s to refer to the mass of steam that appears during the process instead of explicitly being told that all of the water is converted.

Rubric

a) i) Using the specific heat relation $Q = mc\Delta T$ applied to the water (2 points)
   - ii) Correctly writing $m_w c_w (T_f - T_0)$ as this heat (1 point)
   - iii) Using the latent heat relation $Q = mL$ (2 points)
   - iv) Correctly writing $m_s L_v$ as this heat (1 point)
   - v) Reaching the correct answer $Q_w = m_w c_w (T_f - T_0) + m_s L_v$ (1 points)

b) i) Using the specific heat relation $Q = mc\Delta T$ applied to the bowl (2 points)
   - ii) Correctly writing $m_b c_u (T_f - T_0)$ as this heat (1 point)
   - iii) Reaching the correct answer $Q_b = m_b c_u (T_f - T_0)$ (1 point)

c) i) Stating that $Q_c + Q_b + Q_w = 0$ in some form (2 points)
   - ii) Having the correct sign on Q_c (1 point)
   - iii) Using the specific heat relation $Q = mc\Delta T$ applied to the cylinder (2 points)
   - iv) Correctly writing $m_c c_u (T_h - T_f)$ as the magnitude of this heat (1 point)
   - v) Solving correctly for T_h using the solutions provided for (a) and (b) consistent with the sign chosen for Q_c (2 point)
   - vi) Reaching the correct final answer (1 point)

Problem 3

(a) (7 points):

(2 points) Use the ideal gas law to solve for P in terms of V:

$$PV = nRT \to P = \frac{RT_0}{V} \tag{1}$$

since $n = 1$.

- 1 point for using the ideal gas law equation
- 1 point for correctly solving for P in terms of V and constants

(4 points) Set up the work integral:

$$W = \int_{V_0}^{V_1} PdV = \int_{V_0}^{V_1} \frac{RT_0}{V} dV \tag{2}$$

- 1 point for getting the sign right ($dW = +PdV$)
- 1 point for putting the right limits of integration
- 2 points for the correct form of the integral (writing PdV in terms of V and constants

(1 point) Integrate!

$$W = RT_0 \ln\left(\frac{V_1}{V_0}\right) \tag{3}$$

- 1 point for getting the right answer

(b) (8 points):

(2 points) Use the van der Waals equation **??** to solve for P in terms of V:

$$(P + a/V^2)(V - b) = RT \to P = \frac{RT_0}{V - b} - \frac{a}{V^2} \tag{4}$$

- 2 points for correctly solving for P in terms of V and constants

(4 points) Set up the work integral:

$$W = \int_{V_0}^{V_1} PdV = \int_{V_0}^{V_1} \left(\frac{RT_0}{V-b} - \frac{a}{V^2} \right) dV \tag{5}$$

- 1 point for getting the sign right ($dW = +PdV$)
- 1 point for putting the right limits of integration
- 2 points for the correct form of the integral (writing PdV in terms of V and constants

(2 point) Integrate!

$$W = RT_0 \ln \left(\frac{V_1 - b}{V_0 - b} \right) - a \left(\frac{V_1 - V_0}{V_1 V_0} \right) \tag{6}$$

- 2 point for getting the right answer

Problem 4

The maximum efficiency of an engine operating at T_h and T_0 is the carnot efficiency, $e = 1 - \frac{T_0}{T_h}$.

The efficiency of the combined engine is $e = \frac{W_{net}}{Q_h} = \frac{W_1 + W_2}{Q_h}$

We have $W_1 = e_1 Q_h = (1 - \frac{T_c}{T_h})Q_h$. The heat input to the second engine is just Q_c, the heat expelled by the second engine, so $W_2 = (1 - \frac{T_0}{T_c})Q_c = (1 - \frac{T_0}{T_c})\frac{T_c}{T_h}Q_h$. Thus,

$$\begin{aligned} e &= \frac{(1 - \frac{T_c}{T_h})Q_h + (1 - \frac{T_0}{T_c})\frac{T_c}{T_h}Q_h}{Q_h} \\ &= 1 - \frac{T_c}{T_h} + (1 - \frac{T_0}{T_c})\frac{T_c}{T_h} \\ &= 1 - \frac{T_0}{T_h}, \end{aligned}$$

which is the same as the efficiency of a carnot engine operating between T_h and T_0.

Note: An alternate solution is to directly use the formula for the efficiency of a combined engine, derived in a problem in the 7B workbook,

$$e = e_1 + e_2 - e_1 e_2.$$

Using $e_1 = 1 - \frac{T_c}{T_h}$ and $e_2 = 1 - \frac{T_0}{T_c}$ in this equation gives the same result as above.

Rubric

- Correctly states or derives the maximum efficiency of an engine between T_h and T_0, $1 - \frac{T_0}{T_h}$ or $1 - \frac{Q_0}{Q_h}$. (5 points)
- States that the heat input to the second engine is equal to the heat output of the first engine. (2 points)
- Correctly computes the net work done by the first engine. (5 points)
- Correctly computes the net work done by the second engine. (5 points)
- Sets up the total efficiency as $e = \frac{W_1 + W_2}{Q_h}$ (3 points)
- Correctly computes that $e = 1 - \frac{T_0}{T_h}$ (or $1 - \frac{Q_0}{Q_h}$). (5 points)

Alternate Method:

- Correctly states or derives the maximum efficiency of an engine between T_h and T_0, $1 - \frac{T_0}{T_h}$ or $1 - \frac{Q_0}{Q_h}$. (5 points)
- Gives correct expressions for the the efficiency of the two engines or for the net work done by the two engines (2 points)
- States that the efficiency of the combined engine is $e = e_1 + e_2 - e_1 e_2$. (8 points)
- Uses $e = e_1 + e_2 - e_1 e_2$ to correctly compute that $e = 1 - \frac{T_0}{T_h}$ (10 points)

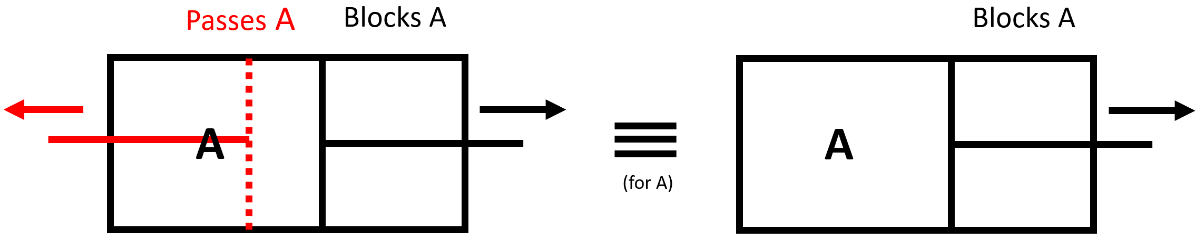

Figure 1: Second process - gas A undergoes a “simple” isothermal expansion.

Problem 5

First, we should think physically about what’s happening. In the first process, we have something like a free expansion of two gasses into a container, and they mix together. This is clearly irreversible - I have to add in a lot of extra effort to put the gases back to their initial state. In the second process, the gas is always in contact with a reservoir at temperature T, so we have isothermal processes for each gas. To help understand what’s happening, I like to draw out the transformation of one of the gases - let’s look at gas A, and note that gas B will behave similarly.

From gas A’s perspective, the partition that lets it through, as well as gas B, may as well not be there. Work done by gas A, heat added to gas A, and internal energy of gas A don’t care about what happens to B, nor do they notice the partition that lets them through. In other words, we can break this process up into what happens to A independently to what happens to B, and add up at the end. The process occuring for A is shown in Figure 1.

We suspect immediately now that second process is reversible - since everything is occuring in thermal equilibrium, I could imagine running the process backwards, and putting back in exactly as much work as I got out when the gas expanded, to put A and B back in their original places. We will see that this suspicion is indeed true later, although you were not asked in this question to discuss this.

1. Beginning with gas A, we have an isothermal (so reversible) process. So

$$\begin{aligned}\Delta S_A =& \int \frac{dQ_{\text{rev}}}{T} = \int \frac{dQ_{\text{isothermal}}}{T} && (7)\\ =&\frac{1}{T}\int dQ \qquad\qquad \text{(T constant here)} && (8)\\ & && (9)\end{aligned}$$

Now for an isothermal process, internal energy is unchanged because the temperature is constant. So by the first law, $dQ = dW = pdV$, so

$$\begin{aligned}\Delta S_A =&\frac{1}{T}\int_{V_A}^{V_A+V_B} pdV && (10)\\ =&\frac{1}{T}\int_{V_A}^{V_A+V_B} \frac{n_ART}{V}dV \qquad \text{(Ideal gas law)} && (11)\\ =&\frac{1}{T}n_ART\ln\left(\frac{V_A+V_B}{V_A}\right) && (12)\\ =&n_AR\ln\left(\frac{V_A+V_B}{V_A}\right). && (13)\end{aligned}$$

Similarly, we find ΔS_B by replacing subscripts A and B in (13). So the total change in entropy of the container is

$$\Delta S_{\text{container}} = n_AR\ln\left(\frac{V_A+V_B}{V_A}\right) + n_BR\ln\left(\frac{V_A+V_B}{V_B}\right). \qquad (14)$$

2. Now this process is irreversible, so one should not simply integrate dQ/T to find ΔS. What one needs to do is find a reversible process that leads to the same outcome, find its change in entropy, and then use that entropy is a state function to find the change in entropy for process 1. But we have just done exactly this! We just calculated the entropy change of process 2, which has the same start and end states as process 1 (gases go from separated to mixed). Since entropy is a state function, we therefore know that

$$\Delta S_{\text{container}} = n_AR\ln\left(\frac{V_A+V_B}{V_A}\right) + n_BR\ln\left(\frac{V_A+V_B}{V_B}\right). \qquad (15)$$

One may be wondering if the end state truly is the same for these two processes.

Remember that for free expansion (where you poke a hole in a container and let the gas spread out), the kinetic energy of each molecule doesn't change, so temperature is unchanged. So the end state of the gas is at the same temperature as the start, only the gases both occupy the full volume and are mixed together. This is exactly the same end state as process 2.

3. Here, one could just notice instantly that process 2 is isothermal, so totally reversible, and therefore conclude that the change in entropy of the universe must be zero. Therefore, the change of entropy of the reservoirs must be minus that of the container, found in (14). To be thorough, I will also show here how to actually calculate it and check this conjecture is correct.

   The reservoir here is isothermally (so reversibly) giving heat to the gas. So its change in entropy is

$$\begin{aligned}\Delta S_{\text{res}} &= \int \frac{dQ_{\text{rev}}}{T} = \int \frac{dQ_{\text{isothermal}}}{T} && (16)\\ &= \frac{1}{T}\int dQ \qquad\qquad \text{(T constant here)} && (17)\\ & && (18)\end{aligned}$$

   Observe that $\int dQ$ is the total heat added to the reservoir. Since any heat added to the reservoir is lost by the gas, we know this is just minus the heat change of the gas. Note that here, the gas gained heat, so we know the reservoir lost heat, so we expect a negative answer.

$$\Delta S_{\text{res}} = \frac{1}{T}\int -dQ_{\text{gas}}. \tag{19}$$

   But we have already found this integral in (8). So we are done here, we can see that the change of entropy of the surroundings is equal and opposite to the change in entropy of the gas.

$$\Delta S_{\text{surroundings}} = -n_A R \ln\left(\frac{V_A + V_B}{V_A}\right) - n_B R \ln\left(\frac{V_A + V_B}{V_B}\right). \tag{20}$$

As a final comment, notice that for process 1, there is no heat exchange with the surroundings. So the surroundings can be thought of as undergoing an isothermal process with zero heat exchange, so the change in entropy of the surroundings is zero. Hence, the change in entropy of the universe in process 1 is the total (14), and so we see process 1 is irreversible.