Second Midterm Solutions

Place all answers on the question sheet provided. The exam is closed textbook/notes/handouts/homework. You are allowed to use a calculator, but not a computer, tablet or smartphone, as well as a two-page cheat sheet. Write all answers clearly and in complete sentences. All answers should be supported by analysis or an argument. This exam has a total of 50 points.

First Name: ______________________________

Last Name: ______________________________

1(a)	1(b)	1(c)	2(a)	2(b)	2(c)	3(a)	3(b)	Total

Honor Code

I resolve

i) not to give or receive aid during this examination, and

ii) to take an active part in seeing that other students uphold this Honor Code.

Signature: ______________________________

1. Visitors arrive to the Japanese Tea Garden in San Francisco according to a Poisson process with rate 100 per hour. Of all the visitors, 40% of them are residents of San Francisco, and the remaining 60% are not. Admission to the garden is \$9 per person for non-residents and \$6 for residents. The garden is open from 9:00 am to 6:00 pm, a total of 9 hours.

   (a) [7 PTS] Compute the expected revenue for one day (9 hours) from visitors to the Japanese Tea Garden.

   Solution:

   Let $N_R(t)$ denote the number of residents visiting the garden during $[0,t]$, and let $N_{NR}(t)$ denote the corresponding number of non-residents. Note that $N_R(t)$ and $N_{NR}(t)$ are independent Poisson random variables with means $100(0.4)t$ and $100(0.6)t$, respectively. The total revenue from visitors is then

$$9N_R(9) + 6N_{NR}(9)$$

   and its expected value is

$$9E[N_R(9)] + 6E[N_{NR}(9)] = 9(100)(0.6)9 + 6(100)(0.4)9 = 7020$$

(b) [7 PTS] Assuming that there were a total of $n = 2k$ visitors to the garden on Monday, what is the probability that there were strictly more residents than non-residents? (Your answer will be in terms of k.)

Solution:

Using the notation from part (a), we want to compute

$$P(N_R(9) \geq k+1 | N_R(9) + N_{NR}(9) = 2k)$$

Now use the fact that conditionally on $\{N_R(9) + N_{NR}(9) = 2k\}$ we have that $N_R(9)$ is Binomial with parameters $(2k, 0.4)$ to compute

$$P(N_R(9) \geq k+1 | N_R(9) + N_{NR}(9) = 2k) = \sum_{i=k+1}^{2k} \binom{2k}{i} (0.4)^i (0.6)^{2k-i}$$

(c) [5 PTS] Of all visitors to the Japanese Tea Garden, 30% are children under 11 years old. Compute the probability that no children under 11 visit the garden between 10:00 am and 1:00 pm.

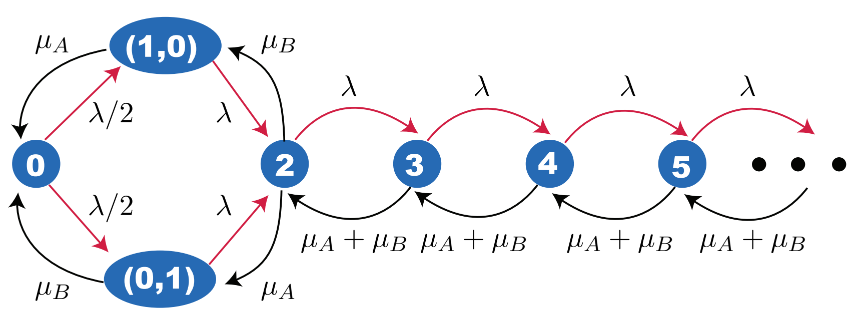
Solution:

Let $N_C(t)$ denote the number of children under 11 who visit the garden during $[0, t]$, and note that $N_C(t)$ is a Poisson random variable with mean $100(0.3)t$. It follows that the probability we want is

$$P(N_C(3) = 0) = e^{-90}$$

2. A local bubble tea store has two people working at the register, call them A and B. A is very efficient and provides service to customers at rate μ_A, while B is slower and provides service at rate $\mu_B < \mu_A$. Customers arrive to the store according to a Poisson process with rate λ, and go straight to a cash register if one is available. All times are measured in minutes. Customers who arrive and find both cashiers busy join a queue, and a customer who finds both cash registers empty choses with equal probability one of the two. Let $X(t)$ the number of customers either at the cash registers or in the queue; all the service times of customers are independent and exponentially distributed (with rates that depend on which cashier services them). We model $\{X(t) : t \geq 0\}$ as a continuous-time Markov chain on the states $\{0, (1, 0), (0, 1), 2, 3, 4, \dots\}$, where state $(1, 0)$ means that there is one customer being served by cashier A, state $(0, 1)$ means that there is one customer being served at cashier B, and state $i = 0, 2, 3, 4, \dots$ means there are a total of i customers in the system.

   (a) [5 PTS] Draw a rate diagram for $\{X(t) : t \geq 0\}$.
   Solution:



   (b) [5 PTS] Write down the rate matrix Q for $\{X(t) : t \geq 0\}$.
   Solution:

$$Q = \begin{bmatrix} -\lambda & \lambda/2 & \lambda/2 & 0 & 0 & 0 & \dots \\ \mu_A & -(\lambda + \mu_A) & \lambda & 0 & 0 & 0 & \dots \\ \mu_B & 0 & -(\lambda + \mu_B) & \lambda & 0 & 0 & \dots \\ 0 & \mu_B & \mu_A & -(\lambda + \mu_A + \mu_B) & \lambda & 0 & \dots \\ 0 & 0 & 0 & \mu_A + \mu_B & -(\lambda + \mu_A + \mu_B) & \lambda & \dots \\ 0 & 0 & 0 & 0 & \mu_A + \mu_B & -(\lambda + \mu_A + \mu_B) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

(c) [7 PTS] At 2:58 pm two friends, Mary and Anne, are standing in queue waiting to get their bubble teas, with Mary standing behind Anne and Anne at the front of the queue. At 3:00 pm Anne starts placing her order with cashier A. How much longer past 3:00 pm can Mary and Anne expect to both be done getting their bubble teas?

Solution:

Note that Mary will start her service once either Anne or the other customer being helped by B is done, and depending on who that is, she'll have to wait for an exponential time to get her bubble tea, call this waiting time W. Call Anne's service time χ_1. Once she starts being served, her service time, call it χ_2 will be exponentially distributed, with a rate that depends on who gets to serve her. Let T denote the time when both will be done. Note that $T = W + \chi_2$ if Mary is served by cashier A, and $T = W + \max\{\chi_1, \chi_2\}$ if Mary is served by cashier B. Conditioning on who finishes first between Anne and the other customer we obtain that

$$
\begin{aligned}
E[T] &= E[W] + E[\chi_2|\text{Anne finishes first}]P(\text{Anne finishes first}) \\
&\quad + E[\max\{\chi_1, \chi_2\}|\text{other customer finishes first}]P(\text{other customer finishes first}) \\
&= \frac{1}{\mu_A + \mu_B} + E[Exp(\mu_A)] \cdot \frac{\mu_A}{\mu_A + \mu_B} \\
&\quad + E[\max\{Exp(\mu_A), Exp(\mu_B)\}] \cdot \frac{\mu_B}{\mu_A + \mu_B} \\
&= \frac{1}{\mu_A + \mu_B} + \frac{1}{\mu_A} \cdot \frac{\mu_A}{\mu_A + \mu_B} \\
&\quad + E[Exp(\mu_A) + Exp(\mu_B) - \min\{Exp(\mu_A), Exp(\mu_B)\}] \cdot \frac{\mu_B}{\mu_A + \mu_B} \\
&= \frac{2}{\mu_A + \mu_B} + \left(\frac{1}{\mu_A} + \frac{1}{\mu_B} - \frac{1}{\mu_A + \mu_B}\right) \frac{\mu_B}{\mu_A + \mu_B} \\
&= \frac{3}{\mu_A + \mu_B} + \frac{\mu_B^2}{\mu_A(\mu_A + \mu_B)^2}
\end{aligned}
$$

3. A small campus coffee shop is operated by a single employee who has a habit of taking "long" breaks from work. In particular, every time he finishes helping a customer and there are no other customers waiting in line he goes to back room to play video games on his phone for a time that is uniformly distributed between 5 and 10 minutes. When he returns from his break he serves customers until there is no one waiting in line, and then immediately takes another break. If upon returning from a break there are no customers waiting for him, he immediately starts a new break. Customers arrive to the coffee shop according to a Poisson process with rate 8 per hour.

   (a) [7 PTS] Compute the mean and variance of the expected number of customers waiting to be served after he returns from one of his breaks.

   Solution:

   Let U be the duration of a break, which is uniformly distributed on $[5, 10]$. Then the number of customers waiting to be served after a break can be written as $N(U)$, where $\{N(t) : t \geq 0\}$ is a Poisson process with rate $8/60 = 2/15$ per minute. Therefore, using the formula for the iterated expectations we get:

$$E[N(U)] = E[E[N(U)|U]] = E[(2/15)U] = (2/15)E[U] = \frac{2}{15} \cdot \frac{5+10}{2} = 1$$

   To compute the variance use the total variance formula to obtain:

$$\begin{aligned} \text{var}(N(U)) &= E[\text{var}(N(U)|U)] + \text{var}(E[N(U)|U]) \\ &= E[(2/15)U] + \text{var}((2/15)U) \\ &= \frac{2}{15} \cdot \frac{5+10}{2} + \left(\frac{2}{15}\right)^2 \cdot \frac{(10-5)^2}{12} \\ &= 1 + \frac{1}{27} \end{aligned}$$

(b) [7 PTS] Compute the probability that there are no customers waiting for him after returning from a break.

Solution:

By conditioning on U we obtain:

$$\begin{aligned} P(N(U)=0) &= \int_5^{10} P(N(U)=0|U=u)\frac{1}{10-5}\,du \\ &= \frac{1}{5}\int_5^{10} P(N(u)=0)du \\ &= \frac{1}{5}\int_5^{10} e^{-(2/15)u}du \\ &= \frac{-3}{2}e^{-(2/15)u}\Big|_5^{10} \\ &= \frac{3}{2}(e^{-2/3}-e^{-4/3}) \end{aligned}$$