Midterm solutions

IND ENG 169 Integer Optimization
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1. ($33.\overline{3}\%$) You recently joined the team of *Kiiwibot* around campus and you would like to use your integer programming skills to improve the food delivery service. At certain point in time, there are N delivery robots and M orders. Each order $m \in \{1, \ldots, M\}$ has a source (restaurant) r_m, a destination (client) c_m, a type (fish, meat, veggies or peanut-free) t_m and a weight of w_m pounds. Let $u_{n,m}$ indicate that order $m \in \{1, \ldots, M\}$ is assigned to robot $n \in \{1, \ldots, N\}$. Model the following restrictions using mixed-integer programming formulations. You may define additional variables if you deem them useful. Formulations are incremental, you don't need to re-state constraints stated in previous answers.

   (a) Each order must be assigned to, at most, one delivery robot.

   (b) Each delivery robot can carry at most K orders, because of volume restrictions, and at most W pounds, because of weight restrictions.

   (c) To prevent cross-contamination with allergens, peanut-free orders can only be transported alone or with other peanut-free orders.

   (d) With all the above restrictions, *Kiiwibot* might not have enough robots to carry all orders. Write an objective function that seeks to minimize the number of unassigned orders.

   Solution.

   (a) $\sum_{n=1}^{N} u_{n,m} \leq 1$ for all $m \in \{1, \ldots, M\}$.

   (b) Volume restriction: $\sum_{m=1}^{M} u_{n,m} \leq K$ for all $n \in \{1, \ldots, N\}$.
   Weight restriction: $\sum_{m=1}^{M} w_m u_{n,m} \leq W$ for all $n \in \{1, \ldots, N\}$.

   (c) Let $M^{\text{peanut-free}} := \{m \in \{1, \ldots, M\} \mid t_m = \text{peanut-free}\}$ and $M^{\text{peanuts}} := \{1, \ldots, M\} \setminus M^{\text{peanut-free}}$. Then, the requirement can be formulated as: $u_{n,i} + u_{n,j} \leq 1$ for all $n \in \{1, \ldots, N\}, i \in M^{\text{peanut-free}}, j \in M^{\text{peanuts}}$.

   (d) Objective: $\min_u \quad \sum_{m=1}^{M} \left(1 - \sum_{n=1}^{N} u_{n,m}\right)$.

2. ($33.\overline{3}\%$) Consider the following linear program:

$$
\begin{aligned}
\min_{x,y,z} \quad & x - 2y + 3z \\
\text{s.t.} \quad & x + 2y + z = 1 \\
& 2x + 3y + 4z = 3 \\
& x + y + 3z = 2 \\
& x \geq 0, y \geq 0, z \geq 0,
\end{aligned}
$$

   for which you know that $(x, y, z) = (0, 1/5, 3/5)$ is a basic optimal solution.

(a) List all other basic feasible solutions.

(b) Is the point $(0, 1/5, 3/5)$ the unique optimal? If you say yes, provide a proof, otherwise, provide another optimal solution.

Solution.

(a) First, note that the 3$^{\text{rd}}$ constraint of the linear program can be obtained by subtracting the 1$^{\text{st}}$ constraint from the 2$^{\text{nd}}$ constraint. Additionally, we know that the problem is feasible (it has an optimal solution). Therefore, the 3$^{\text{rd}}$ constraint can be removed from the linear program without enlarging the feasible set.
We are then left with a linear program with 3 variables (x, y and z) and 2 equality constraints:

$$\begin{aligned} x + 2y + z &= 1 \\ 2x + 3y + 4z &= 3, \end{aligned}$$

which has $\binom{3}{2} = 3$ basic solutions. The basis corresponding to (y, z) gives the basic solution in the statement, with an objective value of $7/5$. The basic solution associated with the (x, y)-basis can be computed by solving

$$\begin{aligned} x + 2y &= 1 \\ 2x + 3y &= 3, \end{aligned}$$

resulting in $(x, y, z) = (3, -1, 0)$. Such basic solution is infeasible.
The basic solution associated with the (x, z)-basis can be, in turn, be computed by solving

$$\begin{aligned} x + z &= 1 \\ 2x + 4z &= 3, \end{aligned}$$

resulting in $(x, y, z) = (1/2, 0, 1/2)$. This is a basic feasible solution with an objective value of 2, completing the list of basic feasible solutions.

(b) The linear program has only two basic feasible solutions, $v = (0, 1/5, 3/5)$ and $w = (1/2, 0, 1/2)$, with v being optimal and w being suboptimal. Because basic feasible solutions correspond to the vertices of the feasible set, any feasible solution different from v and w can be expressed as $\lambda v + (1 - \lambda)w$ for some $\lambda \in (0, 1)$, with an objective value of $\lambda \cdot 7/5 + (1 - \lambda) \cdot 2 = 7/5 + 3/5 \cdot (1 - \lambda) > 7/5$. Therefore, v is the unique optimal solution.

3. ($33.\overline{3}\%$) Consider the following formulations

$$\begin{aligned} F_1 &= \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x + y \leq 4, -1/2 \leq x - y \leq 1/2\} \\ F_2 &= \{(x, y) \in \mathbb{R}^2 \mid \exists z \in [0, 2], -1/8 \leq x - z \leq 1/8, -1/8 \leq y - z \leq 1/8\} \end{aligned}$$

for a certain set $S \subset \mathbb{Z}^2$ (i.e. $S = F_1 \cap \mathbb{Z}^2 = F_2 \cap \mathbb{Z}^2$).

(a) Write S explicitly (i.e. list the points in S).

(b) Draw a sketch of F_1 and F_2.

(c) Show that F_1 is **not better** than F_2 and that F_2 is **not better** than F_1.

Solution.

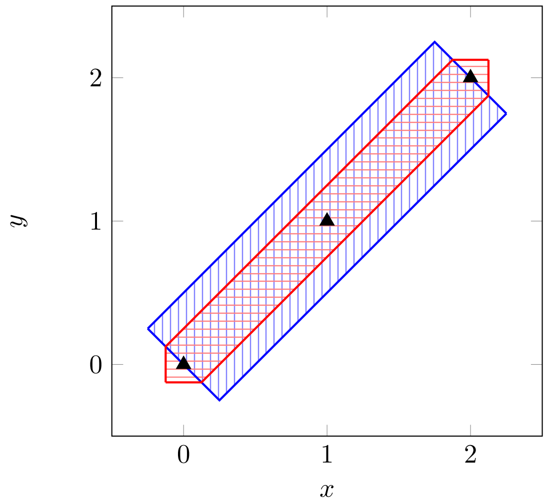

Figure 1: Plot of F_1, F_2 and S for problem 3. F_1 is presented in blue, F_2 in red and the points in S are marked with black triangles.

(a) $S = \{(0,0),(1,1),(2,2)\}$.

(b) A plot overlaying both sets is presented in Fig. 1.

(c) We need to show that (i) $F_1 \not\subseteq F_2$ and (ii) $F_2 \not\subseteq F_1$. For (i) note that $(0, 1/2) \in F_1$, but $(0, 1/2) \notin F_2$, therefore $F_1 \not\subseteq F_2$. For (ii) observe that $(-1/8, -1/8) \in F_2$, but $(-1/8, -1/8) \notin F_1$, therefore $F_2 \not\subseteq F_2$.

4. ($33.\overline{3}\%$) Complete the missing information in the **terminated** enumeration tree of Fig. 2 for an integer linear minimization problem with integer objective coefficients. For missing numerical information, provide any valid value. For missing node status, the possibilities are integer, fractional or infeasible. There should not be contradictions between bounds, pruning, the type of problem and the node status. Nodes are enumerated in the order they are created by branching, and they are evaluated in the same order (i.e. node $i+1$ is evaluated immediately after node i).

Solution.
The enumeration tree of Fig. 2 has been completed in red. Conditions for correct values in fields that admit multiple solutions are enumerated in the following.

- $10 \leq z_1 \leq 14$
- $z_4 \geq 14$
- Node 6:
  - *status* 6 = infeasible, $z_6 = \infty$
  - *status* 6 = fractional, $z_6 \geq 14$
  - *status* 6 = integer, $z_6 \geq 13$
- Nodes 7 and 8:
  - **If** *status* 6 = infeasible or fractional **then** $z_7 \geq 14$, $z_8 \geq 14$.
  - **If** *status* 6 = integer **then** $z_7 \geq \min\{14, z_6\}$, $z_8 \geq \min\{14, z_6\}$.

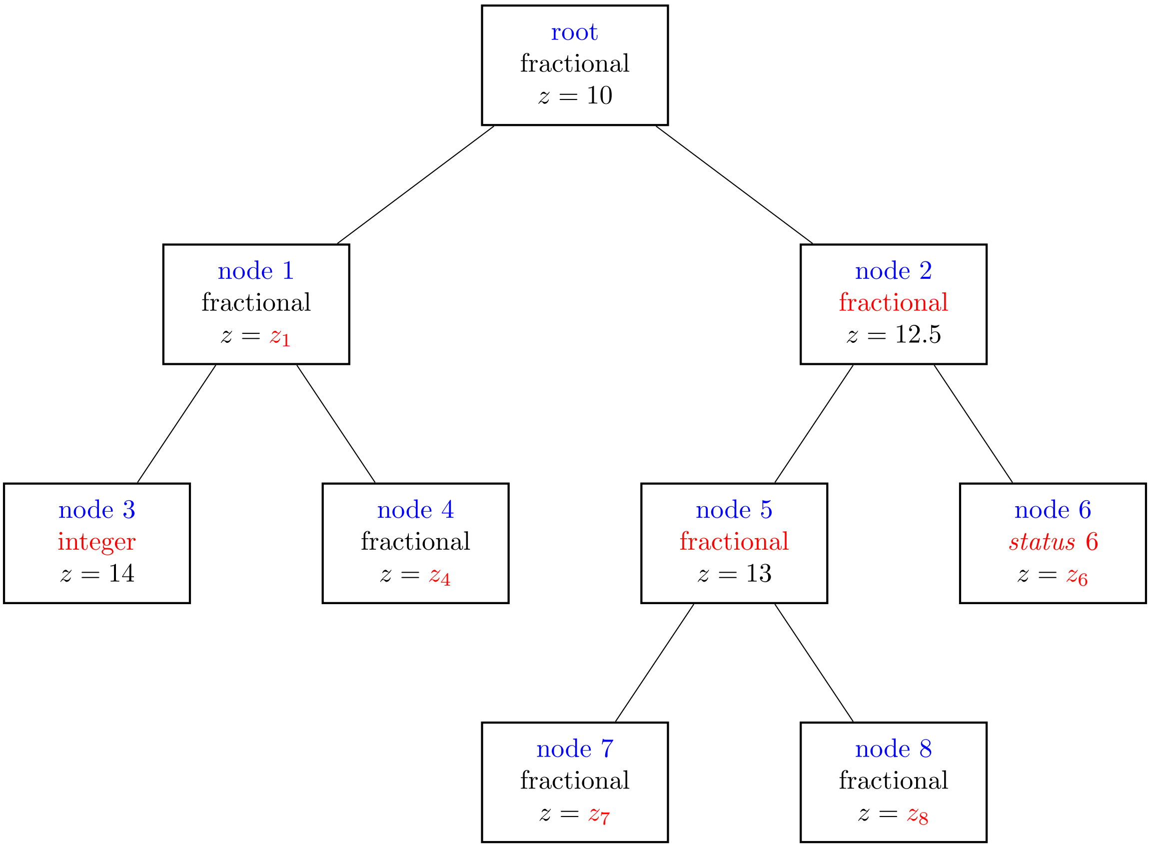

Figure 2: Enumeration tree for problem 4.