PHYSICS 7A, Lecture 1 – Spring 2019
Midterm 2, C. Bordel
Monday, Apr. 1st, 7-9 pm

- Student name:
- Student ID #:
- Discussion section #:
- Name of your GSI:
- Day/time of your DS:

Physics Instructions

All objects in translational motion can be considered as point masses. You may assume that air resistance is negligible (unless specified otherwise) and that the acceleration due to gravity has constant magnitude g close to the surface of the Earth.
Remember that you need to show your work and justify your answers in order to get full credit!

Math Information Sheet

- Solutions to equation $ax^2 + bx + c = 0$ are: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

- $\sin 90° = \cos 0° = -\cos 180° = 1$
- $\sin 0° = \cos 90° = \sin 180° = 0$
- $\sin 45° = \cos 45° = \sqrt{2}/2$
- $\cos (180° - \theta) = -\cos \theta$

- $\sin (180° - \theta) = \sin \theta$
- $\cos (90° + \theta) = -\cos (90° - \theta) = -\sin \theta$
- $\sin (90° + \theta) = \sin (90° - \theta) = \cos \theta$
- $\cos^2 \theta + \sin^2 \theta = 1$

- Circumference of circle: $2\pi R$
- Area of disk: πR^2
- Surface area of sphere: $4\pi R^2$
- Volume of sphere: $4\pi R^3/3$

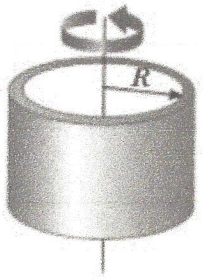
- Volume of cylinder: $\pi R^2 H$
- Lateral area of cylinder: $2\pi RH$
- Arc length: $s = R\theta$
- Volume of cylindrical shell: $2\pi rHdr$

- $(1+x)^a \sim 1+ax$ if $x<<1$
- $\int \frac{dx}{x} = Ln(x) + C$
- $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$

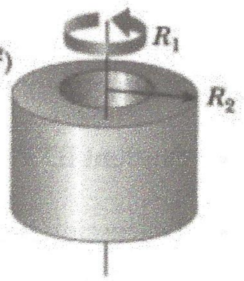
Name:

Rotational inertias

Hoop or cylindrical shell
$I_c = MR^2$

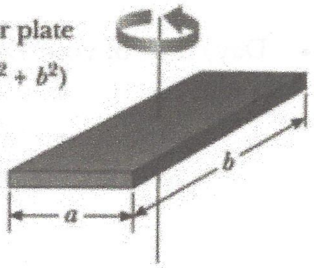


Hollow cylinder
$I_c = \frac{1}{2} M(R_1{}^2 + R_2{}^2)$

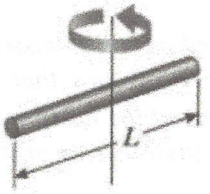


Solid cylinder or disk
$I_c = \frac{1}{2} MR^2$

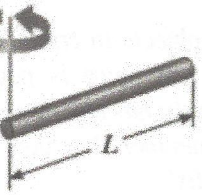
Rectangular plate
$I_c = \frac{1}{12} M(a^2 + b^2)$



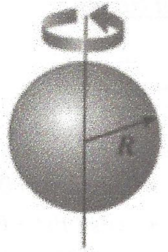
Long thin rod
$I_c = \frac{1}{12} ML^2$



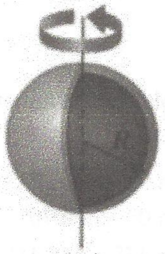
Long thin rod
$I = \frac{1}{3} ML^2$



Solid sphere
$I_c = \frac{2}{5} MR^2$



Thin spherical shell
$I_c = \frac{2}{3} MR^2$



Name:

Problem 1 – Sliding block hitting a spring (25 pts)

A block of mass m is released from rest at the top of an incline forming an angle θ with the horizontal. After traveling a distance L, the block hits a spring of stiffness constant k which is initially in its equilibrium configuration. The surface of the incline is rough above the spring - with a coefficient of kinetic friction μ_k between the block and the ramp - but **smooth under the spring** (Fig.1).

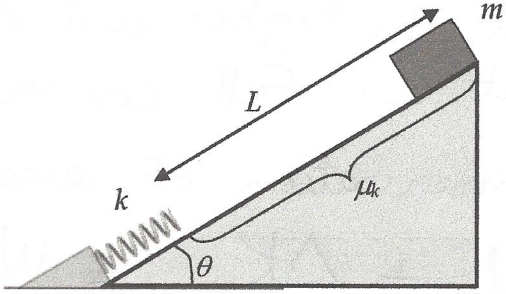
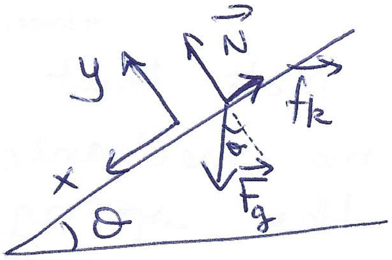


Figure 1

a. Without any calculation, explain whether or not the block reaches the top of the incline after being pushed by the spring.

The existence of kinetic friction between block and ramp results in some mechanical energy being transformed into thermal energy. As a consequence, the block will stop before it can reach the top.



b. Determine the speed of the block when it reaches the spring.

Point A: top of ramp

Point B: upper end of spring

{Block - ramp - spring - Earth} verifies: $\Delta K_{AB} + \Delta U_{AB} = W_{nc}$

Non-conservative forces acting on block are:

- normal force : $W_N = 0$ since $\vec{N} \perp$ displacement (and ramp is static)
- kinetic friction : $W_{f_k} = -\mu_k N L$

Newton's 2nd law projected on y-axis gives $N = mg\cos\theta$

Conservation of energy over A → B:

$\frac{1}{2}\cancel{m}(v_B^2 - v_A^2) - \cancel{m}gH = -\mu_k \cancel{m}gL\cos\theta$ with $H = L\sin\theta$

and $v_A = 0$ since block starts from rest.

then $\boxed{v_B = \sqrt{2gL(\sin\theta - \mu_k\cos\theta)}}$

c. Determine the length of the compression ℓ

Point B: higher end of spring
Point C: full compression

Conservation of energy gives:

$$\Delta U_{BC} + \Delta K_{BC} = W_{nc} = 0 \text{ since } W_N = 0 \text{ (same as b)) and no friction.}$$

$$\Delta U_{BC} = \Delta U_{el} + \Delta U_g = \frac{1}{2}k(\ell^2 - 0) - mg\,\ell\sin\theta$$

then $\frac{1}{2}k\ell^2 - mg\ell\sin\theta + \frac{1}{2}m(\overset{0}{\cancel{v_C^2}} - v_B^2) = 0$

ℓ verifies $\ell = \dfrac{2mg\sin\theta \textcircled{$\pm$} \sqrt{4m^2g^2\sin^2\theta + 4mk\;v_B^2}}{2k}$

We choose $\ell \geqslant 0$

so $\boxed{\ell = \dfrac{2mg\sin\theta + \sqrt{4m^2g^2\sin^2\theta + 8mkgL(\sin\theta - \mu_k\cos\theta}}{2k}}$

d. Determine the distance *d* traveled by the block on the way back, from the point where it loses contact with the spring to where it reaches its maximum height.

Note that v_B is the same on the way back since no mechanical energy is lost to friction under the spring.

Point B: higher end of spring
Point D: max. height on way up

Conservation of energy gives:

$$\Delta U_{BD} + \Delta K_{BD} = W_{nc} = \overset{0}{\cancel{W_N}} + W_{f_k} = -\mu_k N d$$

with $N = mg\cos\theta$

$$\cancel{m}gd\sin\theta + \frac{1}{2}\cancel{m}(\overset{0}{\cancel{v_D^2}} - v_B^2) = -\mu_k d\,\cancel{m}g\cos\theta$$

$$2gd(\sin\theta + \mu_k\cos\theta) = v_B^2$$

Therefore $\boxed{d = \dfrac{\sin\theta - \mu_k\cos\theta}{\sin\theta + \mu_k\cos\theta}\cdot L}$

e. Determine the kinetic friction coefficient that allows the block to get to a stop after traveling a distance $L/2$ on the way back.

$$d = \frac{L}{2} \quad \text{gives} \quad \frac{\sin\theta - \mu_k \cos\theta}{\sin\theta + \mu_k \cos\theta} = \frac{1}{2}$$

$$\text{or} \quad 2\sin\theta - 2\mu_k \cos\theta = \sin\theta + \mu_k \cos\theta$$

$$\sin\theta = 3\mu_k \cos\theta$$

$$\text{then} \quad \boxed{\mu_k = \frac{\tan\theta}{3}}$$

Problem 2 – **L2 Lagrange point (25 pts)**
Lagrange showed that five special points exist in the vicinity of the Earth (mass M_E). They are such that a small satellite of mass m can orbit the Sun (mass M_S) with the same period T as the Earth's. The 2nd Lagrange point, L2, lies on the same radius as the Earth with respect to the center of the Sun, but a small distance d further away from the Sun, as shown in Fig.2. Note that all the other Lagrange points can be ignored in this problem. The distance between the center of the Sun and the center of the Earth is R, and you may assume that $d \ll R$.

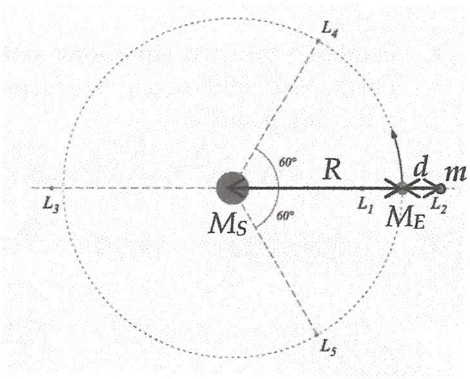


Figure 2

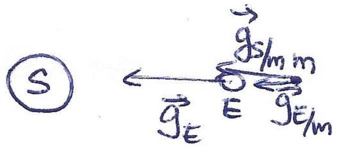
a. Explain why the satellite, located at a different radial distance from the Sun as compared to the Earth, can have the same time period as the Earth's.

For 2 bodies experiencing the same gravitational field, same time period implies same radius of the orbit. However in this case, the satellite experiences the fields generated by Earth and Sun (while Earth is almost exclusively under the action of the Sun since $m \ll M_S$). That makes it possible for satellite and Earth to have same time period.

Name:

b. Determine the magnitude of the gravitational fields $\overrightarrow{g_E}$ and $\overrightarrow{g_m}$ acting on the Earth and satellite, respectively.

- Gravitational field acting on Earth: neglecting the gravitational field due to the satellite, we get: $\boxed{g_E = \frac{GM_S}{R^2}}$



- Gravitational field acting on satellite is vector sum of fields created by Sun and Earth:

$$\boxed{g_m = \frac{GM_S}{(R+d)^2} + \frac{GM_E}{d^2}}$$

c. Establish the two equations satisfied by the time period T for the satellite and the Earth. *Hint: write down the equations of the motion, with the acceleration expressed in terms of the time period T.*

Assuming gravitational force is the only one acting on each body, we have $\vec{F_G} = m\vec{a}$ with $|\vec{a}| = \frac{v^2}{r}$ since motion is circular and uniform.

$v = \frac{2\pi r}{T}$ so $\frac{v^2}{r} = \frac{4\pi^2 r}{T^2}$

- Earth: $\frac{GM_SM_E}{R^2} = \frac{4\pi^2 R M_E}{T^2}$ or $\boxed{\frac{4\pi^2}{T^2} = \frac{GM_S}{R^3}}$

- Satellite: $\frac{GM_Sm}{(R+d)^2} + \frac{GM_Em}{d^2} = \frac{4\pi^2(R+d)m}{T^2}$

then $\boxed{\frac{GM_S}{(R+d)^3} + \frac{GM_E}{(R+d)d^2} = \frac{4\pi^2}{T^2}}$

Name:

d. Using a binomial expansion *(see front page)*, write a first order approximation of the equation satisfied by the time period T for the satellite.

$$\frac{1}{(R+d)^2} = (R+d)^{-2} = R^{-2}\left(1+\frac{d}{R}\right)^{-2} \simeq R^{-2}\left(1-2\frac{d}{R}\right)$$

So satellite verifies:

$$\frac{GM_S}{R^2}\left(1-2\frac{d}{R}\right) + \frac{GM_E}{d^2} = \boxed{\frac{4\pi^2}{T^2}} R\left(1+\frac{d}{R}\right)$$

Substitution using Earth equation

e. Determine the distance d.

$$\frac{GM_S}{R^2}\left(1-2\frac{d}{R}\right) + \frac{GM_E}{d^2} = \frac{GM_S}{R^3} R\left(1+\frac{d}{R}\right)$$

then $$\frac{GM_S}{R^2}\left(1-2\frac{d}{R}-1-\frac{d}{R}\right) = -\frac{GM_E}{d^2}$$

$$\frac{GM_S}{R^2}\left(-\frac{3d}{R}\right) = -\frac{GM_E}{d^2}$$

then $$\boxed{d = \left(\frac{M_E}{3M_S}\right)^{1/3} R}$$

Name:

Problem 3 – Sinking the 9 ball (25 pts)
Both the cue ball (white) and the 9-ball (striped) have identical mass m (Fig. 3). The pool player hits the cue ball in such a way that its initial velocity points along the x-axis and it collides off-center with the 9-ball, initially at rest. After the collision, both balls move away from each other but the 9-ball is equipped with a device that allows its speed (v_9) and direction (ϕ) to be measured. You may ignore the rotational motion of the balls and any frictional force, and assume that the balls are ideally hard.

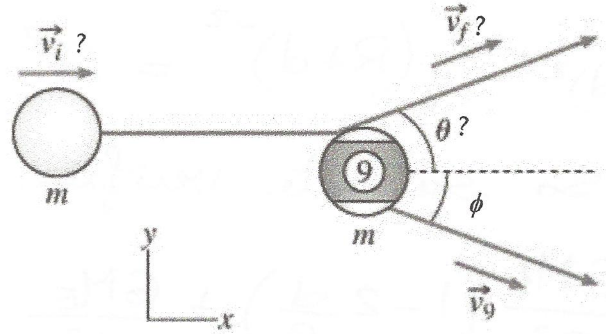


Figure 3

a. Which conservation law(s) are satisfied during the collision and why?

System {cue ball + 9-ball} acted on by gravitational force and normal which exactly cancel each other out so $\vec{F}_{net,ext} = \vec{0}$. Total linear momentum is therefore conserved. In addition, hard objects don't experience deformation so total kinetic energy is also conserved. Collision is perfectly elastic.

b. Determine the angle θ between the direction of the cue ball and the x-axis after the collision.

Momentum conservation: $\cancel{m}\vec{v}_i = \cancel{m}\vec{v}_f + \cancel{m}\vec{v}_9$ (1)

K. Energy conservation: $\frac{1}{\cancel{2}}\cancel{m}v_i^2 = \frac{1}{\cancel{2}}\cancel{m}v_f^2 + \frac{1}{\cancel{2}}\cancel{m}v_9^2$ (2)

$(1)^2$ gives: $v_i^2 = v_f^2 + v_9^2 + 2\vec{v}_f \cdot \vec{v}_9$

$= v_f^2 + v_9^2 + 2v_f v_9 \cos(\theta+\phi)$ Assuming positive angles

Substitution gives $2v_f v_9 \cos(\theta+\phi) = 0$

and therefore $\cos(\theta+\phi) = 0$ assuming non-zero speeds

then $\boxed{\theta = \frac{\pi}{2} - \phi}$

Name:

c. Determine the initial speed v_i of the cue ball.

Projection of linear momentum equation gives:

// x : $v_i = v_f \cos\theta + v_g \cos\phi$ (1)

// y : $0 = v_f \sin\theta - v_g \sin\phi$ (2)

(2) gives $v_f = \frac{v_g \sin\phi}{\sin\theta} = v_g \tan\phi$

then (1) gives : $v_i = v_g \sin\phi \tan\phi + v_g \cos\phi$

$$= v_g \left(\frac{\sin^2\phi}{\cos\phi} + \cos\phi \right)$$

therefore $\boxed{v_i = v_g / \cos\phi}$

d. Determine the final speed v_f of the cue ball.

Eq. (2) from part (c) gives

$$\boxed{v_f = v_g \tan\phi}$$

e. Determine v_i and v_f if, instead, the cue ball had hit the 9-ball head-on.

$\phi = 0$ since collision is one-dimensional

therefore $\boxed{v_f = 0}$ and $\boxed{v_i = v_9}$

according to results found in parts c) and d).

Problem 4 – Rolling cylinder (25 pts)

A solid cylinder of radius d and mass M has non-uniform mass distribution $\rho(r)=kr$, where k is a positive constant and r an arbitrary radial distance measured from the symmetry axis of the cylinder. The cylinder is released from rest at the top of an incline making an angle θ with the horizontal. The cylinder's elevation drops by h from the top to the bottom of the incline. The coefficients of static and kinetic friction between the cylinder and the incline are μ_s and μ_k respectively. You may assume that the cylinder is rolling without slipping, and that its symmetry axis remains perpendicular to the edges of the incline. *Note that d is* ***not*** *negligible compared to h and R.*

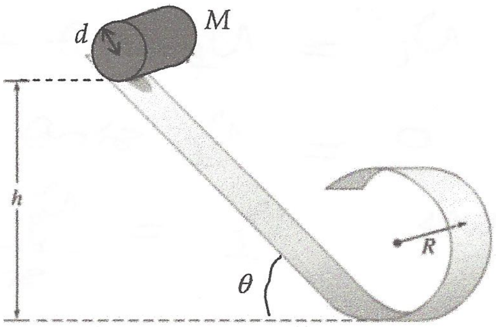


Figure 4

a. Show that the rotational inertia of the cylinder with respect to its symmetry axis is $I = \frac{3Md^2}{5}$. *Hint: assume an arbitrary length for the cylinder and calculate M as well so that you can express I strictly in terms of M and d.*

Mass distribution verifies axial symmetry so we can consider a cylindrical shell of volume

$$dV = 2\pi r H dr = 2\pi r L dr$$ for cylinder of length L.

$$M = \int \rho(r) dV = \int_0^d kr \cdot 2\pi r L dr = 2\pi k L \int_0^d r^2 dr = 2\pi k L \frac{d^3}{3}$$

Rotational inertia about symmetry axis:

$$I = \int dm\, r^2 = \int \rho r^2 dV = \int_0^d kr \cdot r^2 \cdot 2\pi r L dr = 2\pi k L \int_0^d r^4 dr$$

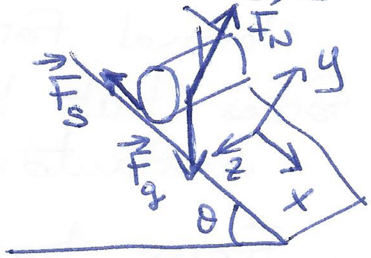
$$= 2\pi k L \frac{d^5}{5}$$

then $$\boxed{I = \frac{3M}{d^3} \cdot \frac{d^5}{5} = \frac{3Md^2}{5}}$$

Name:

b. Determine the acceleration of the center of mass of the cylinder.

- Translation of center of mass with acceleration $\vec{a}$ along the Incline follows: $\vec{F}_{net} = M\vec{a}$



- Rotation of cylinder about axis going through center of mass follows:

$$\tau_{net} = I_{cm}\,\alpha_{cm} \text{ along } z\text{-axis}$$

$//x$: $Mg\sin\theta - F_s = Ma$ (1)

$//y$: $F_N - Mg\cos\theta = 0$ then $F_N = Mg\cos\theta$

$//z$: $-F_s d = \frac{3Md^2}{5}\left(-\frac{a}{d}\right)$ then $F_s = \frac{3M}{5}a$ (2)

Substitution into (1) gives: $\boxed{a = \frac{5g\sin\theta}{8}}$

c. Determine the maximum angle θ that allows the cylinder to avoid slipping.

Eq. (2) from part (b) gives: $F_s = \frac{3M}{\cancel{5}} \cdot \frac{\cancel{5}g\sin\theta}{8}$

No slip condition is $F_s \leq \mu_s F_N$

with $F_N = Mg\cos\theta$ from part (b)

So $\frac{3Mg\sin\theta}{8} \leq \mu_s Mg\cos\theta$

which gives $\tan\theta \leq \frac{8}{3}\mu_s$

$$\boxed{\theta_{max} = \tan^{-1}\left(\mu_s \frac{8}{3}\right)}$$

Name:

d. Determine the total kinetic energy of the cylinder at the top of the loop.

Conservation of energy : $\Delta U + \Delta K = W_{nc}$

Normal force and static friction are non-conservative forces but both apply at point of contact which is instantaneously at rest therefore $W_{F_N} = W_{F_s} = 0$

- From top (A) to bottom (B) of incline :

$$\frac{I}{2}\omega_B^2 + \frac{1}{2}M\left(v_B^2 - \overset{0}{\cancel{v_A^2}}\right) - Mgh = 0 \quad \text{or} \quad K_B = Mgh$$

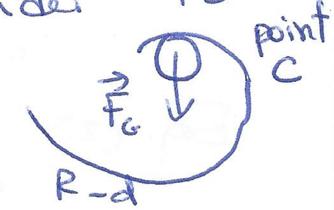
- From B to C (= top of loop) : $(K_C - K_B) + Mg(2R - 2d) = 0$

Substitution gives : $K_C - Mgh + 2Mg(R-d) = 0$

therefore $\boxed{K_C = Mg(h - 2(R-d))}$ Translational + Rotational

e. Determine the minimum height *h* that allows the cylinder to make it through the loop.

We need $F_N \geq 0$ in order for cylinder to remain in contact with track.



Local uniform circular motion with only gravity acting over orbit of radius R-d

gives : $Mg = M\frac{v_c^2}{R-d}$ (1) $\quad v_c$ = min. speed

$$K_{tot}(c) = K_{tr}(c) + K_{rot}(c) = \frac{1}{2}Mv_c^2 + \frac{1}{2}I\omega_c^2 = \frac{M}{2}v_c^2 + \frac{3Md^2}{10}\left(\frac{v_c}{d}\right)^2$$

$$= \frac{8}{10}Mv_c^2$$

So from part (d) we get: $v_c = \sqrt{\frac{5g}{4}\left(h - 2(R-d)\right)}$

Substitution into (1) gives : $g = \dfrac{\frac{5g}{4}\left(h - 2(R-d)\right)}{R-d}$

or $\boxed{h = \frac{14}{5}(R-d)}$ min. height for cylinder to make it through the loop.