# Problem #1

A.

3 points: finding the new value of $r^3$ from the volumetric expansion coefficient

$$V'_{\text{inner sphere}} = \frac{4}{3}\pi(r')^3 = (1 + \beta\Delta T)\frac{4}{3}\pi r^3 \tag{1}$$

$$(r')^3 = (1 + \beta\Delta T)r^3 \tag{2}$$

2 points: solve for r'

$$r' = (1 + \beta\Delta T)^{1/3}r \tag{3}$$

1 point: approximate r' from Taylor series

$$r' \approx (1 + \frac{\beta}{3}\Delta T)r \tag{4}$$

B.

2 points: statements of ideal gas law

$$PV_0 = nR_{\text{gas}}T_i \qquad P'V_f = nR_{\text{gas}}T_f \tag{5}$$

2 points: solving for final pressure

$$P' = P\frac{V_0}{V_f}\frac{T_f}{T_i} \tag{6}$$

2 points: solving for $V_0$ and $V_f$

$$V_0 = \frac{4}{3}\pi(R^3 - r^3) \qquad V_f = \frac{4}{3}\pi(R^3 - (r')^3) = \frac{4}{3}\pi(R^3 - r^3(1 + \beta\Delta T)) \tag{7}$$

2 points: final answer

$$P' = P\frac{R^3 - r^3}{R^3 - r^3(1 + \beta\Delta T)}\frac{T_f}{T_i} \tag{8}$$

C.

1 point: getting rms velocity

$$\overline{\text{KE}}_{\text{trans}} = \frac{3}{2}k_bT = \frac{1}{2}m\overline{v^2} \tag{9}$$

$$v_{rms} = \sqrt{\frac{3k_b T_f}{m}} \tag{10}$$

2 points: getting the rms velocity for motion in the radial direction

$$\overline{KE}_{radial} = \frac{1}{2}k_b T = \frac{1}{2}m\overline{v_{radial}^2} \tag{11}$$

$$v_{radial,\ rms} = \sqrt{\frac{k_b T}{m}} \tag{12}$$

3 points: getting the average time

$$\bar{t} = \frac{l}{v_{radial}} \tag{13}$$

$l =$ the radial distance covered between collisions $= 2(R - r') = 2(R - r(1 + (\beta/3)\Delta T))$

$$\overline{v_{radial}} \approx v_{radial,\ rms} \tag{14}$$

$$\bar{t} = \frac{2(R - r(1 + (\beta/3)\Delta T))}{\sqrt{(k_b T)/m}} \tag{15}$$

# Problem #2

**(a)** (6 pts.) How much time is there between collisions of a particle on a given wall? Because

the gas is isotropic and the side lengths of our container are all equal, the answer will be the same for each wall. Let's consider a wall parallel to the y axis. The distance the particle must travel along the $x$ direction to reach the opposite wall is $L$, the side length. We are interested in the time between successive collisions with the **same** wall, so we need the distance for our particle to travel to the opposite wall and back, which is $2L$.

To find the time needed for the particle to traverse this distance, we consider only the velocity in the $x$ direction, $v_x$, because we have only found the distance traveled in the $x$ direction. Finally we can recall from the definiton of speed/velocity: $v = d/t$

$$\Delta t = \frac{d}{v} = \frac{2L}{|v_x|}$$

Rubric: 2 pts. for correctly identifying the distance traveled as $2L$.

2 pts. for correctly identifying the speed as $v_x$ or $v_y$.

2 pts. for correctly applying the formula for $\Delta t$ in terms of the definition of velocity.

**(b)** (6 pts.) Find an expression for the force $F$ exerted on a given wall by the collision of gas particles. Our starting point is the definition of force:

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$$

During a collision, the velocity of a given particle goes from $\vec{v} = v_x\hat{x} + v_y\hat{y}$ to $\vec{v}' = -v_x\hat{x} + v_y\hat{y}$, so the momentum change is $|\Delta\vec{p}| = |m\Delta\vec{v}| = 2m|v_x|$. The average force $\vec{F}_1$ from a single gas particle is equal to the momentum change from one collision divided by the average time between two collisions.

$$|\vec{F}_1| = \frac{|\delta\vec{p}|}{\delta t} = \frac{mv_x^2}{L}$$

The average force on a wall from collisions of gas particles is equal to the number of particles $N$ times the average value of the average force from one particle:

$$\bar{F} = N\bar{F}_1 = \frac{Nm\bar{v_x^2}}{L}$$

If we want, we can write this in terms of the rms speed $v^2 = v_x^2 + v_y^2 = 2v^2$

$$\bar{F} = N\bar{F}_1 = \frac{Nm\bar{v^2}}{2L}$$

Rubric: 2 pts. for correctly finding the momentum change from a single collision.

2 pts. for correctly identifying the relationship between force and momentum.

2 pts. for correctly using the result of part (a) and finding the average force from all of the particles.

**(c)** (8 pts.) Using the equipartition theorem, derive the equation of state of the gas that relates the temperature $T$, the volume $V$ (correction to area $A$), and the force $F$.

For this problem, we are specifically asked to use the equipartition theorem, and cannot apply the ideal gas law, which is specific to 3 dimensions. Our goal is to *prove* the statement of the ideal gas law in 2D. The equipartition theorem tells us that each accessible quadratic independent degree of freedom in the internal energy of our system has average energy $\frac{1}{2}k_B T$. The kinetic energy in the $x$ direction is one such degree of freedom, so we have the following equation for the average $x$ kinetic energy of a single particle:

$$\frac{1}{2}k_B T = \overline{\frac{1}{2}mv_x^2} = \frac{1}{2}m\bar{v_x^2}$$

or

$$k_B T = \frac{1}{2}mv_{rms}^2$$

The result of part (b) says that $m\bar{v_x}^2 = \frac{FL}{N}$, so we can replace the rms speed to get our equation of state

$$k_B T = \frac{FL}{N}$$

We can simplify this and write in terms of the given variables

$$\boxed{F\sqrt{A} = Nk_B T}$$

Rubric: 3 pts. for a correct application of the equipartition theorem.

3 pts. for eliminating the mass and rms velocity to attain an equation of state.

2 pts. for writing the equation of state in terms of the variables asked for.

# Problem #3

## (a)

Using the ideal gas law:

$$PV = nRT$$

We can write:

$$P_0 V = (1 + 5)RT_i$$

$$\implies \boxed{V = \frac{6RT_i}{P_0}}$$

### Rubric for (a)

- +1pt: Using the ideal gas law

- +1pt: Using $n = 1 + 5 = 6$

- +1pt: Correct final expression

## (b)

Using the ideal gas law again, realizing that the volume doesn't change in any of the processes (we simply have a solid container - there is no mention of the gas being able to do any work on a piston or anything like that) and that $n = 3 + 4 = 7$ after combustion, we can write:

$$P_f V = (3 + 4)RT_f$$

$$\implies P_f = \frac{7RT_f}{V} = \frac{7RT_f}{\frac{6RT_i}{P_0}}$$

$$\implies \boxed{P_f = \frac{7T_f P_0}{6T_i}}$$

## Rubric for (b)

- +1pt: Using the ideal gas law

- +1pt: Realizing that the volume does not change from (a)

- +1pt: Using $n = 7$ at the end of combustion

- +1pt: Correct final expression

## (c)

Using the expression for $P_f$ from (b) and the fact that we are given that $T_f = T_i$ we can write:

$$P_f = \frac{7T_f P_0}{6T_i} = \frac{7T_f P_0}{6T_f}$$

$$\implies \boxed{P_f = \frac{7P_0}{6}}$$

## Rubric for (c)

- +1pt: Using the ideal gas law in some way

- +2pt: Plugging in $T_i$ for $T_f$ and simplifying the expression

- +1pt: Correct final expression

## (d)

We are given that every species has 5 degrees of freedom in *each* step. Recall that:

$$E_{int} = \frac{d}{2}Nk_B T = \frac{d}{2}nRT$$

where $d$ is the number of degrees of freedom. Since both $n$ and $T$ are changing from just before to just after the combustion in the problem (we are not considering the cooling process in this part), we look at the initial and final values of $E_{int}$:

$$E_{int_{initial}} = \frac{5}{2}(6)RT_i$$

$$E_{int_{final}} = \frac{5}{2}(7)RT_f$$

Then we can look at the change in internal energy of the system ($\Delta E_{int}$) when undergoing the combustion:

$$\Delta E_{int} = E_{int_{final}} - E_{int_{initial}} = \frac{5}{2}(7)RT_f - \frac{5}{2}(6)RT_i$$

$$\implies \boxed{\Delta E_{int} = \frac{5}{2}R(7T_f - 6T_i)}$$

## Rubric for (d)

- +1pt: Realizing that the coefficient in the expression for internal energy is $\frac{5}{2}$

- +1pt: Realizing that $Nk_B = nR$ and using $nR$ in the expression for internal energy

- +1pt: Showing or using $\Delta E = E_{int_{final}} - E_{int_{initial}}$, not just $\Delta E = \frac{5}{2}nR\Delta T$, which is only true if $n$ is constant

- +1pt: Correct final expression

## (e)

Since the volume doesn't change for any of the processes, no work is done. Then, by the first law of thermodynamics:

$$E_{int} = Q - W = Q$$

In this part, we are only looking at the cooling process, so $n$ is now constant (specifically $n = 7$), and we can use a similar procedure as in part (d) to find the change in internal energy (and therefore the heat that leaves the container):

$$E_{int_{initial}} = \frac{5}{2}(7)RT_f$$

$$E_{int_{final}} = \frac{5}{2}(7)RT_i$$

$$\implies \Delta E_{int} = E_{int_{final}} - E_{int_{initial}} = \frac{5}{2}(7)RT_i - \frac{5}{2}(7)RT_f$$

$$\implies Q = \frac{35}{2}R(T_i - T_f)$$

$$\implies \boxed{-Q = \frac{35}{2}R(T_f - T_i)}$$

Where $-Q$ is the heat that leaves the container. Adding up the values in (d) and (e), we see that the internal energy increases overall because, even though we have reached the same temperature as the initial temperature, the number of moles has increased.

## Rubric for (e)

- +1pt: Showing or clearly working with the fact that $Q = \Delta E_{int}$

- +1pt: Working on (*only*) the cooling process and not the earlier combustion process

- +2pt: Mentioning that the reason the answers in parts (d) and (e) don't correspond is that the number of moles (or number of molecules) change in the combustion process but don't change in the cooling process for the same change in temperature, and therefore the magnitudes of the two answers are different

- +1pt: Correct final expression

# Problem #4

(a) [8pt] See below for an acceptable example. It's important to draw the line shape correctly (e.g. distinct from isovolumetric/isobaric), and the direction of the cycle should be right (engine instead of a fridge). You should also include sufficient descriptions to show which process is isothermal or adiabatic (to convince others that it's not a random drawing).
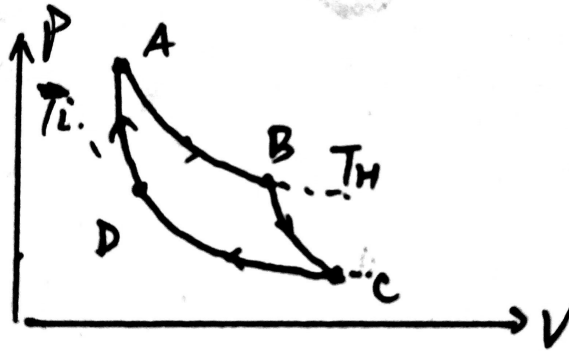


(b) [17pt] There are two possible ways to solve the problem: expression $Q_H$ and $Q_L$ with either work or entropy.

**Solution 1** According to first law: $W = Q_H - Q_L$. Using $\Delta E = 0$ in isothermal process, we have $Q_H = W_H$, $-Q_L = W_L$. so [3pt]

$$e = W/Q_H = 1 - Q_L/Q_H = 1 + W_L/W_H \tag{16}$$

AB is isothermal process, so [3pt]

$$W_H = \int_A^B p(V)dV = \int_A^B \frac{nRT_H}{V}dV = nRT_H \ln\frac{V_B}{V_A} \tag{17}$$

Similarly for CD: [3pt]

$$W_L = nRT_L \ln\frac{V_D}{V_C} \tag{18}$$

Meanwhile, since BC is adiabatic process: [2pt]

$$p_B V_B^\gamma = p_C V_C^\gamma \quad (\gamma = \frac{c_p}{c_v} > 1) \tag{19}$$

And the same holds for DA: [2pt]

$$p_A V_A^\gamma = p_D V_D^\gamma \tag{20}$$

Using the fact from isothermal process: $p_A V_A = p_B V_B$, $p_C V_C = p_D V_D$, we have [2pt]

$$\frac{p_B V_B^\gamma \cdot p_A V_A}{p_A V_A^\gamma \cdot p_B V_B} = \frac{p_C V_C^\gamma \cdot p_D V_D}{p_D V_D^\gamma \cdot p_C V_C} \implies \left(\frac{V_B}{V_A}\right)^{\gamma-1} = \left(\frac{V_C}{V_D}\right)^{\gamma-1} \tag{21}$$

In the end we obtain that

$$\frac{V_B}{V_A} = \frac{V_C}{V_D} \implies \ln\frac{V_B}{V_A} = -\ln\frac{V_D}{V_C} = C \tag{22}$$

Therefore: [2pt]

$$e = 1 + \frac{W_L}{W_H} = 1 + \frac{-nRT_L C}{nRT_H C} = 1 - \frac{T_L}{T_H} \tag{23}$$

**Solution 2** According to first law: $W = Q_H - Q_L$, so

$$e = W/Q_H = 1 - Q_L/Q_H \tag{24}$$

and we can use entropy change $\Delta S$ to represent $Q_L$ and $Q_H$. [3pt]

AB is isothermal process: [3pt]

$$\Delta S_{AB} = \int_A^B dQ/T_H = Q_H/T_H \tag{25}$$

Similarly for CD: [3pt]

$$\Delta S_{CD} = -Q_L/T_L \tag{26}$$

BC and AD are adiabatic process: [2pt]

$$\Delta S_{BC} = \Delta S_{DA} = 0 \tag{27}$$

The system return to its original state after a cycle: [3pt]

$$\Delta S_{ABCDA} = 0 \tag{28}$$

Input the $\Delta S$ values, we end up with [1pt]

$$Q_H/T_H - Q_L/T_L = 0 \implies Q_L/Q_H = T_L/T_H \tag{29}$$

Therefore: [2pt]

$$e = 1 - \frac{Q_L}{Q_H} = 1 - \frac{T_L}{T_H} \tag{30}$$

# Problem #5

## Correct entropy change – 12.5 points

The overall entropy change is $\Delta S = \Delta S_{\text{Cs}} + \Delta S_{\text{room}}$, where the former can be further broken up into $S_{\text{warming}}$ and $S_{\text{melt}}$. Correctly adding all of these $\frac{11m}{3}$ J/K. Hence, getting all the components right but getting this final answer wrong (either due to an arithmetic mistake or needlessly rounding within the problem) can be regarded as equivalent to getting this "almost correct", corresponding to the 0.5 point deduction.

## Correct entropy change from warming up – 4 points

This is the entropy change due to the cesium warming up after melting, and should result as $12m$ J/K.

### Correct setup – 3 points

We have that $\dot{S} = \frac{\dot{Q}}{T}$, while the heat required to warm up a material is given by $\dot{Q} = mc\dot{T}$, so:

$$\Delta S = mc \int_{T_0}^{T_f} \frac{\dot{T}}{T} = mc \ln \left( \frac{T_f}{T_0} \right) \tag{31}$$

Where we would use that the temperatures are given in Kelvins (i.e. $300\,\text{K}$ and $320\,\text{K}$ for $T_0$ and $T_f$ respectively), and $c = 200\,\text{J}/\text{kg}\cdot\text{K}$ as given in the problem, to get the correct answer. Note that substituting in numerical values to begin with and not explicitly having the symbolic expression is fine – correct form but incorrect numerical values would then garner these points but none further.

## Almost correct entropy change from warming up – 3.5 points

This is given in the case of some algebraic error, or mistake in converting symbols to numbers. It is not given in the case that the answer was kept as a purely symbolic expression (as numerics are necessary for comparison) or in the case of a more fundamental

mistake between the setup and final answer, i.e. not converting Celcius to Kelvins. All subsequent "almost correct" items are similar, so I will not repeat this explanation.

## Correct entropy change from melting – 4 points

This should come out to $\frac{200m}{3}$ J/K.

### 0.0.1 Correct setup – 3 points

This part is quite simple: as temperature is constant, $\Delta S = \frac{Q}{T}$, where $Q = mL$, where $L = 20000$ J/kg as given in the problem, and $T = T_0 = 300$ K.

## Almost correct entropy change from melting – 3.5 points

## Correct entropy change for room – 4 points

This should come out to $-75m$ J/K, where mistaking just the sign would lead to being marked as having "correct setup".

### Correct setup – 3 points

For here, we note that (due to the implicit assumption that the room with cube is an isolated system) energy conservation guarantees us that $Q_{\text{room}} = -Q_{\text{cube}}$. Meanwhile, the room stays at a constant temperature equal to $T_f$. Hence the expression here is:

$$\Delta S = -\frac{m\left(L + c\left(T_f - T_0\right)\right)}{T_f} \tag{32}$$

## Almost correct entropy change for room – 3.5 points

# Process will occur naturally – 0.5 points

Self-explanatory: the process will indeed occur naturally.

# Correct reasoning why – 2 points

Since $\Delta S > 0$ for the overall system, this is a process that could occur naturally (i.e. second law of thermodynamics). Alternatively (again due to the second law), one could say that heat naturally flows from the hot room to the cold cesium until the two reach the same temperature. Points will be awarded as long as reasoning is correct here, even if answer is wrong (ex. due to wrong entropy calculation).

# Bonus

In 3 dimensions, the Maxwell distribution is

$$f(v) = 4\pi N \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right)$$

What would the Maxwell distribution look like if there were 5 dimensions instead of 3? Explain any changes that you made from the 3D version of the Maxwell distribution.

If you have some knowledge of statistical mechanics, it is possible to simply perform a calculation and derive an expression for the Maxwell distribution in $d$ dimensions. Alternatively, you may remember Professor Birgeneau discussing it in lecture. Here we'll use heuristic arguments to determine how each term should change. First let's analyze the exponent, which is simply the kinetic energy divided by $k_B T$. The formula for kinetic energy will be the same in 5D, though we'll have $v^2 = v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2$ now. This part of the formula comes from the Boltzmann distribution and will be unchanged.

Now consider the $v^2$ term. We know that velocities in 1 dimension follow a normal distribution. In higher dimensions, we add a factor to account for the size of phase space. There are more ways to have a speed of $v_2$ than $v_1$ for $v_2 > v_1$ because the sphere in velocity space defined by $\vec{v}^2 = v_2^2$ is bigger. Heuristically and in the formal derivation, the $v^2$ factor comes from the surface area of a ball in $d$ dimensions, which scales as $v^{d-1}$. Therefore we change $v^2 \to v^4$.

Finally we have to consider the normalization constant. We can calculate this explicitly even without doing the derivation by setting

$$\int_0^\infty dv f(v) = 1$$

We can guess what happens to the dimensionful constants without going through this process just with dimensional analysis, however. The units of $f$ are inverse speed, as are the units of $\sqrt{\frac{m}{k_B T}}$. Therefore we need to add two more factors of $\sqrt{\frac{m}{k_B T}}$ in order to cancel out the added $v^2$. It's hard to guess the numerical prefactor, but it's natural to change the $3/2$ to a $5/2$ without touching anything else. Our guess is then

$$f(v) = 4\pi N \left(\frac{m}{2\pi k_B T}\right)^{5/2} v^4 \exp\left(-\frac{mv^2}{2k_B T}\right)$$

The actual answer with full normalization is

$$f(v) = \frac{8\pi^2}{3} N \left(\frac{m}{2\pi k_B T}\right)^{5/2} v^4 \exp\left(-\frac{mv^2}{2k_B T}\right)$$

where $\frac{8\pi^2}{3}$ is related to the surface area of a ball in 5 dimensions.

Rubric: 5pts. for correctly changing the exponents of $v$ and not $m/k_B T$ while not changing the argument of the exponential. Must be accompanied by some justification.