EE 120: Signals and Systems
Midterm 2
November 10, 2018

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Problem 1 (75 Points) [Dereverberation] In this problem well explore aspects of the phenomenon of *reverberation* and the signal processing of *dereverberation*.

Reverberation of a signal x is the superposition of the signal with delayed and weighted copies of itself. Reverberation is used by the music industry to enhance the voices of vocal artists. Performance halls, too, are often acoustically designed to reverberate the voices of artists and the sounds of musical instruments during performances.

In this problem well stay entirely in the discrete-time realm. In particular, we'll consider the reverberation model

$$\begin{aligned}\forall n \in \mathbb{Z},\ \ y(n) &= x(n) + \alpha x(n-N) + \alpha^2 x(n-2N) + \alpha^3 x(n-3N) + ... \\ &= \sum_{\ell=0}^{\infty} \alpha^{\ell} x(n - \ell N)\end{aligned}$$

where $0 < \alpha < 1$ represents an attenuation factor that could be due to reflection of the input signal x from a barrier; $N \in 1, 2, 3, ...$ denotes the fundamental delay in samples; and y is the output signal representing the reverberated version of x.

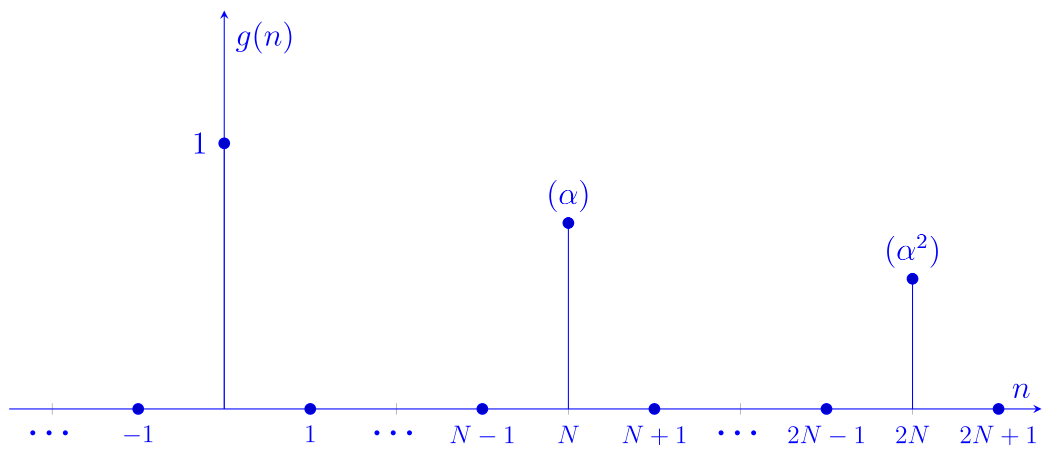
The reverberation model is well-represented by a causal, BIBO-stable DT-LTI system G having frequency response $G(\omega)$ and impulse response $g(n)$.

(a) Determine a reasonably simple expression for, and provide a well- labeled plot of, the impulse response $g(n)$.

Solution:
Plugging in $x(n) = \delta(n)$ and $y(n) = g(n)$, we have:

$$g(n) = \sum_{\ell=0}^{\infty} \alpha^{\ell} \delta(n - \ell N) = \begin{cases} \alpha^{\ell} u(\ell), & \ell \bmod N = 0 \\ 0, & \text{otherwise} \end{cases}$$

So the impulse response is a right-sided decaying exponential of rate α up-sampled by a factor of N. The decay occurs because $0 < \alpha < 1$ (note that we are guaranteed decay because α is strictly less than 1), and the plot must reflect the decaying nature of the signal:



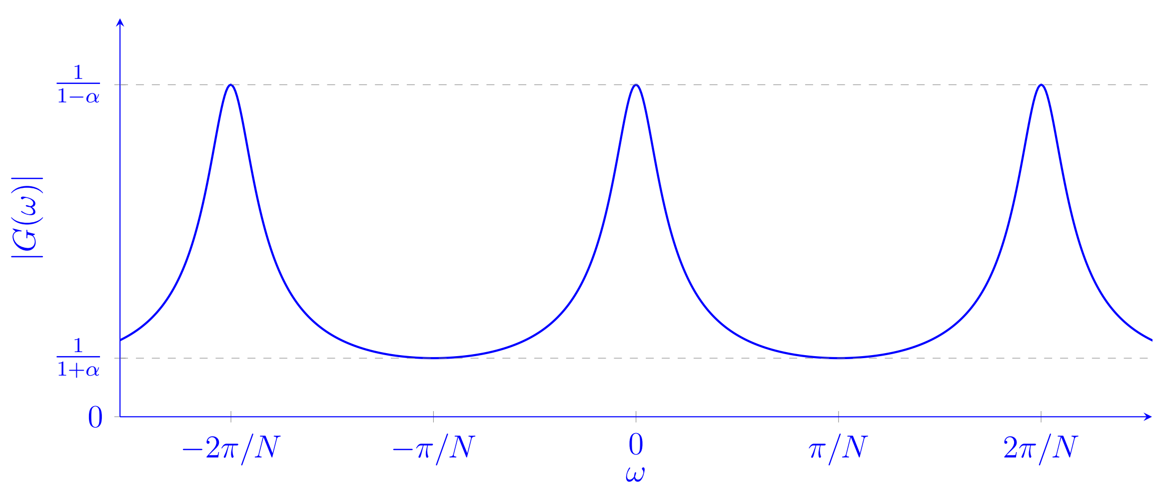
(b) (20 points) Determine a reasonably simple expression for the frequency response $G(\omega)$, and provide a well-labeled plot of the magnitude response $|G(\omega)|$. You may tackle this part independently of the previous one.

Solution:

From the analysis equation,

$$
\begin{aligned}
G(\omega) &= \sum_{n=-\infty}^{+\infty} g(n) e^{-i\omega n} & (1) \\
&= \sum_{k=0}^{+\infty} \alpha^n e^{-i\omega k N} & (2) \\
&= \frac{1}{1-\alpha e^{-i\omega N}} & (3)
\end{aligned}
$$

where (1) is the DTFT analysis equation, (2) is due to (a), and (3) is the formula for the infinite sum of a geometric series, which we may apply since $0 < \alpha < 1$ is given in the problem.



(c) (15 points) The input-output behavior of the system G is described by the following linear, constant-coefficient difference equation:

$$y(n) = \beta y(n-\gamma) + x(n-\mu)$$

where $\beta \in \mathbb{R}$, $\gamma, \mu \in \mathbb{Z}$. Determine β, γ, μ in terms of the known parameters α, N.

Solution:
At this point, we can read off the inverse Fourier transform of the frequency response of the filter G.

$$G(\omega) = \frac{1}{1-\alpha e^{-i\omega N}} \implies y(n) - \alpha y(n-N) = x(n)$$

Reorganizing to match the given form, we can then read off the required parameters:

$$\begin{aligned} & y(n) = \alpha y(n-N) + x(n) \\ \implies & \beta = \alpha \\ & \gamma = N \\ & \mu = 0 \end{aligned}$$

(d) In some contexts reverberation is undesirable—for example, if the delay N is too long. In this and other scenarios we want to devise system to eliminate reverberation. In particular, we want to design a DT-LTI system H such that when its placed in series with G, we can recover the original signal x from y.

(i) Determine a reasonably simple expression for, and provide a well-labeled plot of, the impulse response $h(n)$ of the system H.

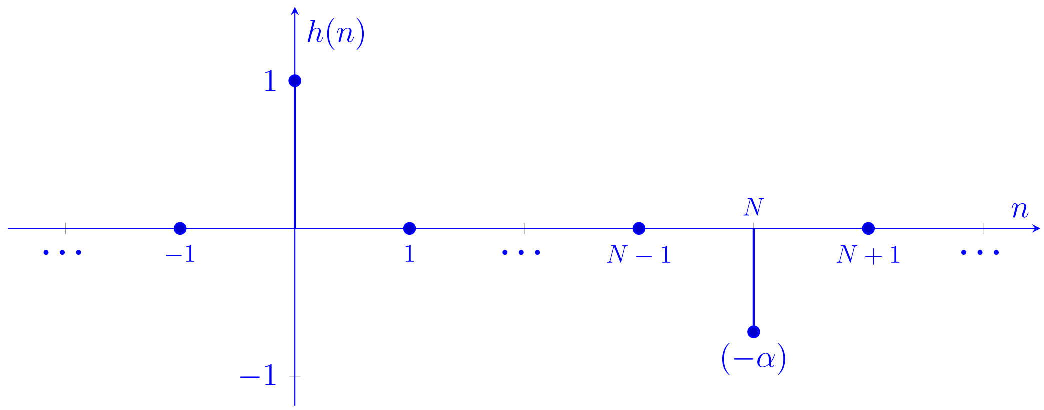
Solution:
We use the inverse filter equation:

$$\begin{aligned} H(\omega)\,G(\omega) &= 1 \\ H(\omega) &= \frac{1}{G(\omega)} \\ H(\omega) &= 1 - \alpha e^{-i\omega N} \end{aligned}$$

Now, we pattern match to obtain the impulse response. We know the Fourier transform pair from our formula sheet: $\delta(n-N) \xleftrightarrow{\mathcal{F}} e^{-i\omega N}$. Therefore:

$$h(n) = \delta(n) - \alpha\delta(n-N)$$

Here's a plot:



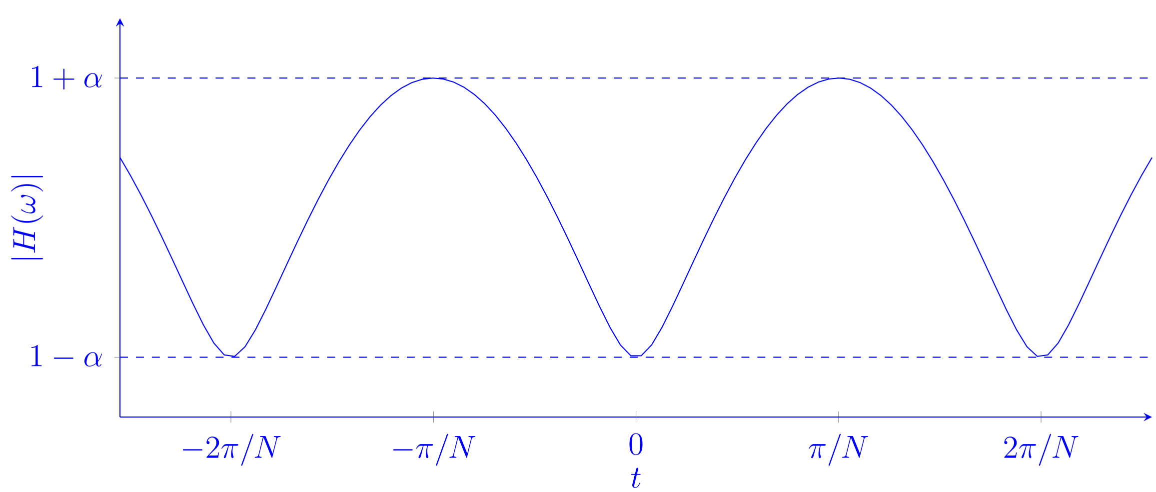
(ii) Determine a reasonably simple expression for the frequency response $H(\omega)$, and provide a well-labeled plot of the magnitude response $|H(\omega)|$ of the inverse system H.

Solution:

We see from part d(i) that:

$$H(\omega) = 1 - \alpha e^{-i\omega N}$$
$$|H(\omega)| = |e^{i\omega N}(1 - \alpha e^{-i\omega N})| = |e^{i\omega N} - \alpha|$$

The second statement is true because phase shifts do not change the magnitude. Drawing this on a unit circle, we see that the maximum and minimum value are obtained when $e^{i\omega N}$ and $-\alpha$ are colinear. The minimum value is obtained at $\omega = 0$, $|H(0)| = 1 - \alpha$. The maximum value is obtained at $\omega = \pi/N$, $|H(\pi/N)| = 1 + \alpha$. We also see that $H(\omega)$ is $2\pi/N$ peridic, resulting in the following plot:



Problem 2 (95 points) The frequency response of a CT-LTI filter H is given by

$$\forall\, \omega \in \mathbb{R},\, \exists\, \sigma > 0 : \quad H(\omega) = e^{-\sigma|\omega|}$$

(a) (15 points) Show that the impulse response $h(t)$ of the filter is of the form

$$\forall\, t \in \mathbb{R},\, \exists\, A, B > 0, \quad h(t) = \frac{A}{B^2 + t^2}$$

and determine reasonably simple expressions for the constant parameters A and B in terms of σ.

Solution:

$$\begin{aligned}
h(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega t}\, d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\infty} e^{-\sigma\omega}e^{i\omega t}\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{0} e^{\sigma\omega}e^{i\omega t}\, d\omega \\
&= \frac{1}{2\pi}\left[\int_{0}^{\infty} e^{\omega(it-\sigma)}\, d\omega + \int_{-\infty}^{0} e^{\omega(it+\sigma)}\, d\omega\right] \\
&= \frac{1}{2\pi}\left[\left.\frac{e^{\omega(it-\sigma)}}{it-\sigma}\right|_{0}^{\infty} + \left.\frac{e^{\omega(it+\sigma)}}{it+\sigma}\right|_{-\infty}^{0}\right] \\
&= \frac{1}{2\pi}\left[-\frac{1}{it-\sigma} + \frac{1}{it+\sigma}\right] \\
&= \frac{1}{2\pi}\frac{-2\sigma}{-\sigma^2 - t^2} = \frac{\frac{\sigma}{\pi}}{\sigma^2 + t^2}
\end{aligned}$$

From which we read off

$$\begin{aligned}
A &= \frac{\sigma}{\pi} \\
B &= \sigma
\end{aligned}$$

(b) (15 points) Provide a well-labeled plot of $h(t)$ and determine the values of t at which $h(t) \leq \frac{1}{2}h(0)$.

Solution:

$$h(0) = \left.\frac{\frac{\sigma}{\pi}}{\sigma^2 + t^2}\right|_{t=0} = \frac{1}{\sigma\pi}$$

We are then looking to find the set

$$\mathcal{T} = \left\{t \in \mathbb{R} \,:\, h(t) \leq \frac{1}{2\sigma\pi}\right\}$$

Expanding and simplifying this inequality, we have

$$\begin{aligned}\frac{\frac{\sigma}{\pi}}{\sigma^2 + t^2} &\le \frac{1}{2\sigma\pi} \\ t^2 + \sigma^2 &\ge 2\sigma^2 \\ t^2 &\ge \sigma^2\end{aligned}$$

Many students failed to obtain the set $\mathcal{T}$ from here. We must realize that $\mathcal{T}$ is not a connected set, and is in fact the union of two disjoint sets:

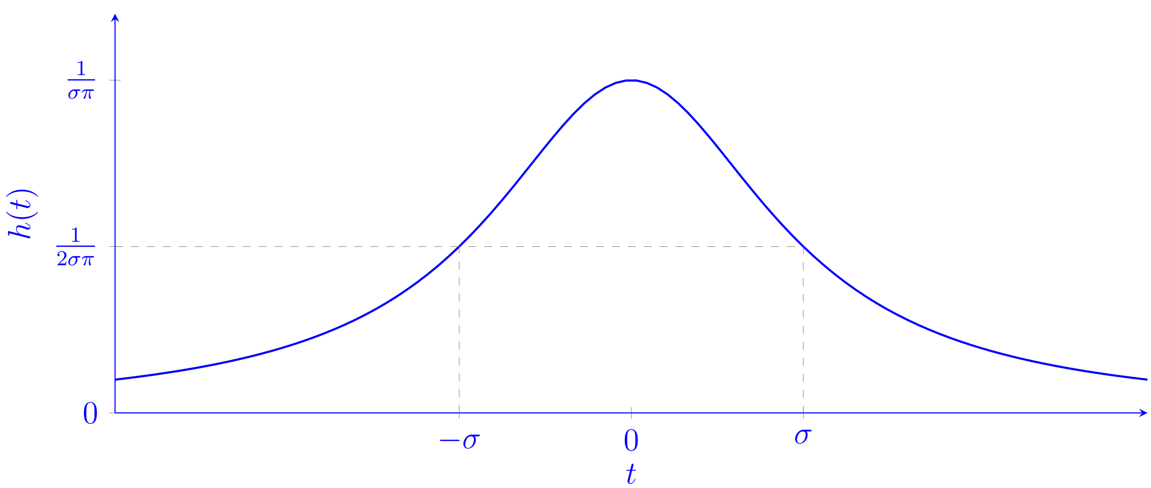
$$\mathcal{T} = \{t \in \mathbb{R} : t \le -\sigma\} \cup \{t \in \mathbb{R} : t \ge \sigma\}$$

We can write this concisely as

$$\mathcal{T} = \{t \in \mathbb{R} : |t| \ge \sigma\}$$

For the plot, points were awarded for the following, with points awarded generously for error carried forward:

- Correct shape, shown below.
- $h(0) = \frac{1}{\sigma\pi} = \frac{A}{B^2}$



(c) (5 points) Select the strongest true statement from the following:

(a) The filter H *must* be causal.

(b) $\boxed{\text{The filter } H \text{ } \textit{cannot} \text{ be causal.}}$

(c) We have insufficient information to determine whether the filter H is causal.

Provide a succinct, but clear and convincing, explanation for your selection.
Solution:
It is clear from the plot in the previous part that $h(t) \neq 0,\ \forall\, t > 0$.

(d) (15 points) Select the strongest true statement from the following:

(a) $\boxed{\text{The filter } H \textit{ must} \text{ be BIBO stable.}}$

(b) The filter H *cannot* be BIBO stable.

(c) We have insufficient information to determine whether the filter H is BIBO stable.

Provide a succint, but clear and convincing explanation for your selection. If you choose (a) or (b), then you must, as part of your explanation, evaluate

$$\int_{-\infty}^{+\infty} |h(t)|\, dt$$

Solution:
To show that the filter must be BIBO stable, we demonstrate that the impulse response is absolutely integrable:

$$\int_{-\infty}^{+\infty} |h(t)|\, dt = \int_{-\infty}^{+\infty} \left| \frac{\frac{\sigma}{\pi}}{\sigma^2 + t^2} \right| dt \tag{4}$$

$$= \int_{-\infty}^{+\infty} \frac{\frac{\sigma}{\pi}}{\sigma^2 + t^2} dt \tag{5}$$

$$= \int_{-\infty}^{+\infty} \frac{\frac{\sigma}{\pi}}{\sigma^2 + t^2} e^{-i0t} dt \tag{6}$$

$$= H(0) = e^{-\sigma|0|} = 1 \tag{7}$$

Where (2) follows from the nonnegativity of $h(n)$, (3) follows from the fact that $e^{-i0t} = 1,\ \forall\, t$, and (4) follows from the analysis equation.

(e) (20 points) For this part only, suppose we apply the input signal

$$\forall t \in \mathbb{R}, x(t) = \frac{sin(t)}{\pi t}$$

to the filter H. Determine a reasonably simple expression for

$$E_y = \int_{-\infty}^{+\infty} |y(t)|^2 dt$$

the energy of the corresponding output signal y.

Solution:
Note that:

$$X(\omega) = \begin{cases} 1 & |\omega| \leq 1 \\ 0 & else \end{cases}$$

In general, we have:

$$Y(\omega) = H(\omega)X(\omega)$$

and so

$$Y(\omega) = \begin{cases} e^{-\sigma|\omega|} & |\omega| \leq 1 \\ 0 & else \end{cases}$$

Parseval's Theorem states

$$\int_{-\infty}^{+\infty} |y(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |Y(\omega)|^2 d\omega$$

So we compute

$$E_y = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |Y(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-1}^{1} |H(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-1}^{1} |e^{-\sigma|\omega|}|^2 d\omega = \frac{1}{2\pi} \int_{-1}^{1} e^{-2\sigma|\omega|} d\omega$$

$$= \frac{1}{2\pi}\Big(\int_{-1}^{0} e^{2\sigma\omega} d\omega + \int_{0}^{1} e^{-2\sigma\omega} d\omega \Big) = \frac{1}{2\pi}\Big(\frac{e^{2\sigma\omega}}{2\sigma}\Big|_{-1}^{0} - \frac{e^{-2\sigma\omega}}{2\sigma}\Big|_{0}^{1} \Big)$$

$$= \frac{1}{2\pi}\Big(\frac{1 - e^{-2\sigma}}{2\sigma} - \frac{e^{-2\sigma} - 1}{2\sigma} \Big) \qquad = \frac{1}{2\pi\sigma}\Big(1 - e^{-2\sigma}\Big)$$

Note that $|Y(\omega)|^2$ is an even function, so we can jump from

$$\frac{1}{2\pi} \int_{-1}^{1} e^{-2\sigma|\omega|} d\omega$$

straight to

$$\frac{1}{2\pi} 2 \int_{0}^{1} e^{-2\sigma\omega} d\omega$$

for an easier alternative way to solve the integral.

(f) (25 Points) For this part only, suppose we apply the standard impulse train as the input signal to the filter H. That is, let

$$\forall t \in \mathbb{R} \text{ and } \exists T > 0, \quad x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

(a) (15 points) Provide a well-labeled plot of $Y(\omega)$, the spectrum of the corresponding output signal $y(t)$.
We know $Y(\omega) = H(\omega)X(\omega)$, and $H(\omega)$ is given, so we just need to find $X(\omega)$.

Since $x(t)$ is a T-periodic signal, it has a CTFS represenation. For convenience, let $\omega_0 = 2\pi/T$ denote the fundamental frequency. Then, we have

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t} \tag{8}$$

$$X_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-ik\omega_0 t}\mathrm{d}t \tag{9}$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-ik\omega_0 (0)}\mathrm{d}t \tag{10}$$

$$= \frac{1}{T} \tag{11}$$

$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{ik\omega_0 t} \tag{12}$$

In general, for a fixed $\Omega \in \mathbb{R}$,

$$e^{i\Omega t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega - \Omega) \tag{13}$$

Applying this CTFT pair to (12),

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \tag{14}$$

$$y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} e^{-\sigma|\omega|}\delta(\omega - k\omega_0) \tag{15}$$

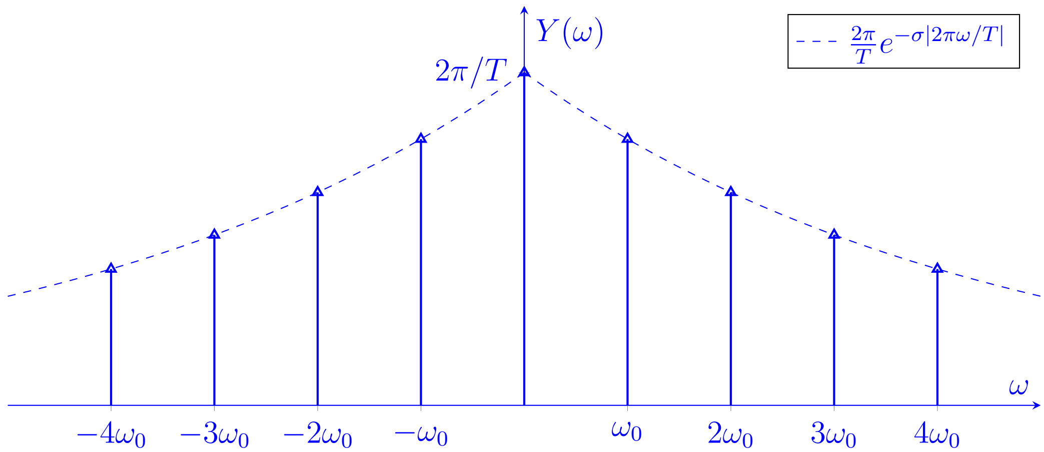
$$= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} e^{-\sigma|k\omega_0|}\delta(\omega - k\omega_0) \tag{16}$$

$$= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} e^{-\sigma|2\pi k/T|}\delta\left(\omega - k\frac{2\pi}{T}\right) \tag{17}$$

A common mistake was taking the CTFT of $x(t)$ directly to get

$$X(\omega) = \sum_{n=-\infty}^{\infty} e^{-i\omega nT} \tag{18}$$

The CTFT synthesis and analysis equations will only converge to useful values for absolutely-integrable or square-integrable signals. However, it turns out that signals with CTFS representations do have Fourier transforms, regardless of the aforementioned summability conditions.



(b) (10 Points) Since the input x to the filter is periodic, so is the output y. Determine a reasonably simple expression for the coefficients Y_k in the CTFS expansion of y:

$$y(t) = \sum_{k=-\infty}^{+\infty} Y_k e^{i2\pi kt/T}$$

You may tackle this part independently of part (i), but your results in the two parts must be consistent.

Solution 1: $x(t)$ is a linear combination of complex exponentials, so we can apply the eigenfunction property of LTI systems.

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{ik\omega_0 t} \tag{19}$$

$$y(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} H(k\omega_0) e^{ik\omega_0 t} \tag{20}$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T} e^{-\sigma|2\pi k/T|}}_{Y_k} e^{ik\omega_0 t} \tag{21}$$

Solution 2: Apply the synthesis equation to retrieve $y(t)$ from $Y(\omega)$.

$$\begin{aligned}
y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{i\omega t}\mathrm{d}\omega && (22)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\pi}{T}\sum_{k=-\infty}^{\infty} e^{-\sigma|2\pi k/T|}\delta\left(\omega - k\frac{2\pi}{T}\right)\right)e^{i\omega t}\mathrm{d}\omega && (23)\\
&= \sum_{k=-\infty}^{\infty}\frac{1}{T}e^{-\sigma|2\pi k/T|}\int_{-\infty}^{\infty}\delta\left(\omega - k\frac{2\pi}{T}\right)e^{i\omega t}\mathrm{d}\omega && (24)\\
&= \sum_{k=-\infty}^{\infty}\frac{1}{T}e^{-\sigma|2\pi k/T|}e^{i2\pi kt/T}\int_{-\infty}^{\infty}\delta\left(\omega - k\frac{2\pi}{T}\right)\mathrm{d}\omega && (25)\\
&= \sum_{k=-\infty}^{\infty}\underbrace{\frac{1}{T}e^{-\sigma|2\pi k/T|}}_{Y_k}e^{i2\pi kt/T} && (26)
\end{aligned}$$

Problem 3 (20 points) Consider a BIBO stable DT-LTI filter H that has frequency response $H(\omega)$ and a *real-valued* impulse response $h(n)$ each of which is known. We apply the input signal

$$\forall\, n \in \mathbb{Z}, \quad x(n) = \cos(\omega_0 n)$$

to the filter. Show that the corresponding response is

$$\forall\, n \in \mathbb{Z}, \quad y(n) = |H(\omega_0)|\cos\left(\omega_0 n + \angle H(\omega_0)\right)$$

Solution:
We begin with the standard approach of recognizing that the input consists of two frequencies, which can be considered to be scaled separately by an LTI system:

$$x(n) = \frac{e^{i\omega_0 n + e^{-i\omega_0 n}}}{2} \implies y(n) = H(\omega_0)\frac{e^{i\omega_0 n}}{2} + H(-\omega_0)\frac{e^{-i\omega_0 n}}{2}$$

Motivated by our observation that the desired expression for the output is in terms of $H(\omega_0)$, we seek to express $H(-\omega_0)$ in terms of $H(\omega_0)$. Here, the fact that $h(n)$ is real-valued comes in handy, allowing us to apply conjugate symmetry. Additionally, the form of the final expression suggests a phase-magnitude decomposition:

$$H(-\omega_0) = H^*(\omega_0) = |H(\omega_0)|\, e^{-i\angle H(\omega_0)}$$
$$H(\omega_0) = |H(\omega_0)|\, e^{i\angle H(\omega_0)}$$

Substituting this back into our expression for $y(n)$, we get

$$\begin{aligned}
y(n) &= |H(\omega_0)|\left(\frac{e^{i\omega_0 n + \angle H(\omega_0)} + e^{-i\omega_0 n - i\angle H(\omega_0)}}{2}\right)\\
&= |H(\omega_0)|\cos\left(\omega_0 n + \angle H(\omega_0)\right)
\end{aligned}$$