Mathematics 54.1
Midterm 1, 4 October 2019
50 minutes, 50 points

NAME: ______________________ **ID:** ______________________

GSI: ____

INSTRUCTIONS:
Justify your answers, except when told otherwise.
All the work for a question should be on the respective sheet.

This is a CLOSED BOOK examination, NO NOTES and NO CALCULATORS are allowed.
NO CELL PHONE or EARPHONE use is permitted.
Please turn in your finished examination to your GSI before leaving the room.

Q1	/15
Q2	/14
Q3	/8
Q4	/13
Tot	

Question 1. (15 points) FTTFF FFFFT FTTTT TFTFF

○ The dimensions of the nullspace and the left nullspace of any $m \times n$ matrix A agree.

○ If a set S of vectors spans $\mathbb{R}^n$, then S contains at least n vectors.

○ The map $T : \mathbb{R}^3 \to \mathbb{R}^3$ which reflects points about the plane $x + y + z = 0$ is a linear transformation.

○ The first column of AB is the first column of the matrix A multiplied on the right by B.

○ If one row in the echelon form of the augmented matrix of a system is $[0\,0\,0\,1\,1]$, then the system is inconsistent.

○ The solution set of a consistent $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$ is a linear subspace of $\mathbb{R}^n$.

○ The rows of any 4×5 matrix are linearly dependent.

○ The determinant of a square matrix is the product of its pivots.

○ For any two $n \times n$ matrices A, B, we have $(AB)^T = A^T B^T$.

○ If the linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent, then the coefficient matrix A does not have a pivot position in every row.

○ The kernel of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the left nullspace of A.

○ The determinant of a change-of-coordinates matrix cannot be zero.

○ For invertible $n \times n$ matrices A, B, C, we have $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

○ If a set of vectors in $\mathbb{R}^n$ is linearly independent, then it contains at most n vectors.

○ If the coefficient matrix A has a pivot position in every column, then the system $A\mathbf{x} = \mathbf{b}$ has at most one solution.

○ If the matrix A is invertible, then the inverse of A^{-1} is A itself.

○ A linear combination of a collection of vectors in $\mathbb{R}^5$ is a linear subspace.

○ If a linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$, then the reduced row echelon form of A is I_n.

○ For any two square matrices A, B of the same size, $\det(A + B) = \det(A) + \det(B)$.

○ If the collection $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of vectors in $\mathbb{R}^5$ is linearly dependent, then at least one of the vectors is a multiple of one of the other two.

Question 2. (4+8+2 points)
All questions pertain to the matrix A below. In all cases, make sure that your methods are clear.
(a) Is the system $A\mathbf{x} = [1, 1, 4]^T$ consistent? If so, find a particular solution.
(b) Find bases of the nullspace, column space, row space and left nullspace.
(c) For what values of a, b does $[4, 7, a, b]$ lie in the row space?
Check your work! Small mistakes can cost you many points in this question.

$$A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 5 & 3 & 2 \\ 5 & 14 & 9 & 5 \end{bmatrix}$$

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The particular solution with free variables 0 is $[-2, 1, 0, 0]^T$.

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{Nul} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \text{Col} \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 14 \end{bmatrix}; \text{Row : top two rows of rref; LNul}[3, 1, -1].$$

Need $a = 3, b = 4$.

Question 3. (2+3+3 points)
The coordinate vector of $\mathbf{v} \in \mathbb{R}^2$ with respect to the basis $\mathcal{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ of $\mathbb{R}^2$ is: $[\mathbf{v}]_\mathcal{B} = \begin{bmatrix} a \\ 1 \end{bmatrix}$.
(a) What is the vector $\mathbf{v}$ in the standard basis?
(b) Find the coordinate vector $[\mathbf{v}]_{\mathcal{B}'}$ of $\mathbf{v}$ with respect to the basis $\mathcal{B}' = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. (Explain.)
(c) Represent graphically the bases $\mathcal{B}, \mathcal{B}'$ as they appear in standard coordinates, as well as the set of all vectors $\mathbf{v}$, as a ranges over all real values.

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$$\text{standard } \mathbf{v} = \begin{bmatrix} a+3 \\ 2a+4 \end{bmatrix}; \text{rref} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

so the change of coordinates matrix is the last 2×2 block, new coordinates for $\mathbf{v}$ are $a - 1$ and 2. The line of vectors is $y = 2x - 2$.

Question 4. (5+3+5 points)
(a) By using determinants, find the values of $k \in \mathbb{R}$ for which the matrix A is invertible.
(b) Find $\det(A^{-1})$ when A is invertible. (*Hint:* you don't need to compute A^{-1}.)
(c) Find the inverse matrix for $k = 1$ by your favorite method. Check your answer.

$$A = \begin{bmatrix} k & 1 & 2 \\ k^2+1 & 3 & 4 \\ 2k & 2 & 3 \end{bmatrix}$$

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$$\det(A) = 9k + 8k + 4k^2 + 4 - 12k - 8k - 3k^2 - 3 = k^2 - 3k + 1, \quad \text{so } k \neq \frac{3}{2} \pm \frac{\sqrt{5}}{2}.$$

$$\det(A^{-1}) = \det(A)^{-1} \text{ from product formula } \det(A)\det(A^{-1}) = \det(AA^{-1}) = 1$$

$$A^{-1} = \begin{bmatrix} -1 & -1 & 2 \\ -2 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \quad \text{at } k = 1$$