EECS 126 Probability and Random Processes — UC Berkeley, Fall 2019
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Midterm 1

Last Name	First Name	SID

Rules.

- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- You have 10 minutes to read the exam and 90 minutes to complete it.
- The exam is not open book. No calculators or phones allowed.
- Unless otherwise stated, all your answers need to be justified. Show your work to get partial credit.
- Maximum you can score is 120 but 100 points is considered perfect.
- Don't read too much into the size of the answer boxes.

Problem	points earned	out of
Problem 1		49
Problem 2		15
Problem 3		19
Problem 4		17
Problem 5		20
Total		120

1 Assorted Problems [7+7+7+7+7+7+7]

(a) **MGF Basics**

Let X be a random variable whose MGF is given as:

$$M_X(t) = \frac{1}{6}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{4}e^{2t}$$

Compute the probability that $|X| \leq 1$.

(b) **Weather Entropy**

Let W take on values in $\{sunny, clear, rainy\}$, denoting the weather for a specific day. It is *sunny* on 50% of days, *clear* on 25% of days, and *rainy* on 25% of days. Now, let C take on values in $\{tshirt, jacket, sweater\}$, denoting the clothing that Justin is wearing for a specific day. Suppose Justin chooses his clothes based on the weather W of the day, with probabilities given by the table below; for example, when it is *sunny*, he wears a *tshirt* 50% of the time. What is the entropy of C?

Clothing	*sunny*	*clear*	*rainy*
tshirt	50%	0%	0%
jacket	50%	25%	75%
sweater	0%	75%	25%

(c) **Gaussian Comparison**

In this problem we will consider normal RVs $\mathcal{N}(\mu, \sigma^2)$. Let X_1 be $\mathcal{N}(0,1)$, X_2 be $\mathcal{N}(2,25)$, and X_3 be $\mathcal{N}(\text{-}1,100)$. Rank the following probabilities from least to greatest. Your answer should be in the form of ? <? <?.

$$a = P(0 < X_1 < 1) \qquad b = P(4 < X_2 < 7) \qquad c = P(-6 < X_3 < 4)$$

(d) **Random Functions**

Let the set S_N denote the first N positive integers, i.e. $\{1, 2, 3, \ldots, N\}$. We randomly generate a function f over S_N as follows. For each $x \in S_N$, let $f(x)$ be uniform in S_N. Let X be the size of the range of f, i.e. $X = |\{f(x) : x \in S_N\}|$.

Example: for $N = 2$, the uniform sample space of f and the corresponding values of X would be:

$$\begin{aligned}
f_1 &: f_1(1) = 1, f_1(2) = 1, X = 1\\
f_2 &: f_2(1) = 1, f_2(2) = 2, X = 2\\
f_3 &: f_3(1) = 2, f_3(2) = 1, X = 2\\
f_4 &: f_4(1) = 2, f_4(2) = 2, X = 1
\end{aligned}$$

Compute $\mathbb{E}[X]$.

(e) **3x3x3 Black Cube**

Suppose you have a 3x3x3 inch cube of solid black marble. You paint all 6 faces white and chop it up into 1x1x1 inch subcubes. You then toss all of these in the bag, and with your eyes closed, you take one out and roll it. Opening your eyes, you notice that 5 faces are black. What's the probability that the face you can't see (i.e. on the bottom) is also black?

(f) **1s, 0s, and -1s**

Given a list containing exactly n 1's, m -1's, and k 0's, where $n > m$, what is the probability that the sum of each prefix is strictly positive (not including the empty list)?

Example: If our list is $[1, 0, 1, -1, 0, 1, -1]$, then we are looking for the event that $[1]$ and $[1, 0]$ and $[1, 0, 1]$ and ... and $[1, 0, 1, -1, 0, 1, -1]$ all are positive when you sum their elements (which in this case is true).

(g) **Ants**

There are n ants uniformly and independently distributed on a number line from 0 to 1. Each ant, with equal probability and independent of other ants, moves left or right at a speed of 1 unit per second. When two ants hit each other, they bounce back in opposite directions. An ant finishes once it reaches either 0 or 1. How long does it take for the last ant to finish in expectation?

Hint: Think carefully about two ants colliding. Is it as complicated as it seems?

2 L&S CS and EECS [7+8]

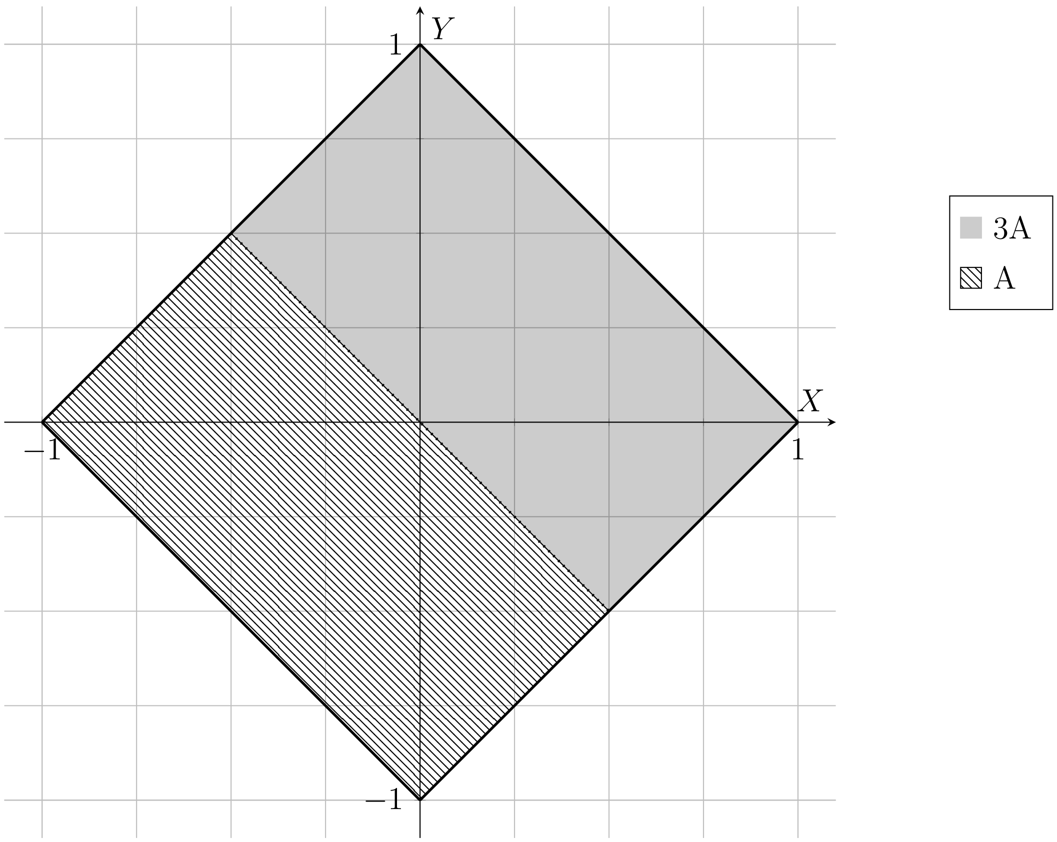
There are n L&S CS students and n EECS students sitting at a round table. Every seating permutation is equally likely.

(a) What's the probability that all the L&S CS students sit next to each other?

(b) Let X be the number of lonely EECS students (i.e. EECS students who are not sitting next to any other EECS students). What is $\mathbb{E}[X]$?

3 Graphical Diamond [3+8+8]

Random variables X and Y have a joint density as pictured below:



(a) Find the value of A.

(b) Find $f_Y(y)$ for $y \in [0, 1]$ (Note: this is not a condition, just a way to reduce the amount of computation you have to do).

(c) Find $\mathbb{E}[X]$.

4 Laplace [8+9]

Suppose $X \sim \text{Laplace}(0, b)$, where the PDF for the zero-centered Laplace distribution is

$$p_X(x) = \frac{1}{2b} \exp(-\frac{|x|}{b})$$

(a) What's the conditional PDF for X if we know that $X > k$, where $k > 0$?

Hint: The Laplace distribution is a composition of symmetric Exponentials.

(b) Compute the variance of $X \sim \text{Laplace}(0, b)$. Your answer should depend on b.

5 NumPy Confusion [5+7+8]

NumPy uses a method called inverse transform sampling to generate a number according to a random variable X. In particular, given the CDF F_X of X, they generate a seed $U \sim \text{Uniform}[0,1]$ and then return $Z = F_X^{-1}(U)$.

(a) Show that X and Z have the same CDF.

(b) Suppose $X \sim \text{Exponential}(1)$. What is the corresponding Z in terms of U?

(c) In NumPy, calling `np.random.exponential(0,λ)` actually generates a number according to the random variable $\text{Exponential}(\frac{1}{\lambda})$, not $\text{Exponential}(\lambda)$. Suppose we passed in the rate λ instead of the mean $\frac{1}{\lambda}$. We meant to generate according to $X \sim \text{Exponential}(\lambda)$, but instead did according to $Y \sim \text{Exponential}(\frac{1}{\lambda})$. What is $E\left[\left(F_X^{-1}(U) - F_Y^{-1}(U)\right)^2\right]$?

Hint: Use part (b).