First Midterm Examination
Closed Books and Closed Notes

Question 1
A Planar Oscillator (25 POINTS)

As shown in Figure 1, a particle of mass m is attached to a point A by a linear spring spring of unstretched length ℓ_0 and stiffness K and is free to move on a smooth vertical plane. A vertical gravitational force $-mg\mathbf{E}_y$ also acts on the particle. The point A is given a prescribed vertical motion:

$$\mathbf{r}_A = f(t)\mathbf{E}_y, \tag{1}$$

where $f(t)$ is a known function of time.

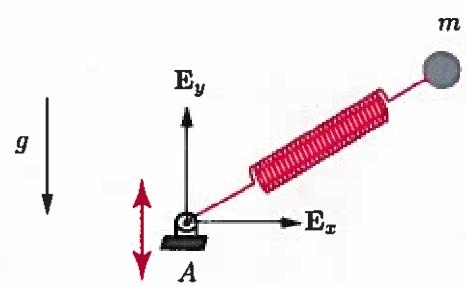


Figure 1: *A particle of mass m that is attached to a point A by a linear spring of stiffness K and unstretched length ℓ_0. The particle is free to move on a smooth vertical plane.*

(a) *(6 Points)* Starting from the representation

$$\mathbf{r} = r\mathbf{e}_r + f(t)\mathbf{E}_y, \tag{2}$$

establish representations for the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ vectors of the particle.

(b) *(4 Points)* Draw a free-body diagram of the particle. For full credit, provide a clear expression for the spring force.

(c) *(10 Points)* Show that the differential equations governing the motion of the particle are

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -K\,(?) - m\left(g + \ddot{f}\right)\sin(\theta),$$
$$m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = -m\left(g + \ddot{f}\right)\cos(\theta). \tag{3}$$

For full credit, supply the missing term.

(d) *(5 Points)* Suppose the motion of A has a constant velocity. With the help of the equations of motion (3) show that is is possible for the particle to remain at rest relative to A if

$$\theta = -\frac{\pi}{2}, \qquad r_0 = \frac{mg}{K} + \ell_0. \tag{4}$$

Question 2 *A Waterslide Design* (30 Points)

As shown in Figure 2, a preliminary design for a water slide has a section featuring a hump. To model the slide and test the efficacy of the slide, we model humans as particles of mass m in motion on a rough curve subject to a gravitational force. The portion of the slide of interest is modeled as a cubic curve:

$$\mathbf{r} = x\mathbf{E}_x + \left(x - \frac{c}{3}x^3\right)\mathbf{E}_y, \tag{5}$$

where $c > 0$ is a constant with dimensions of meters^{-2}.

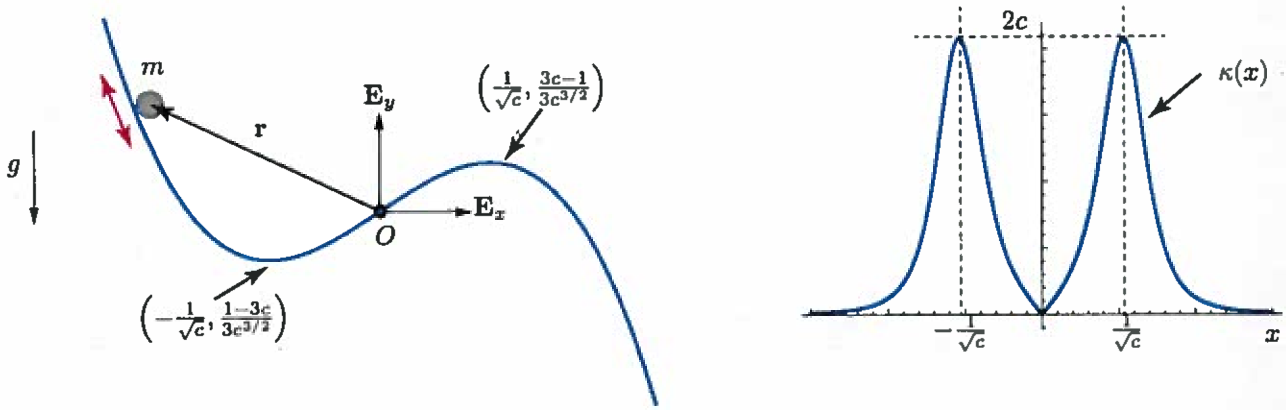


Figure 2: *Schematic of a particle of mass m which is free to move on a rough plane curve. A graph of the curvature $\kappa = \kappa(x)$ is also shown. For the results shown in this figure, the constant $c > 0$.*

(a) *(6 Points)* Assuming that the particle is moving on the curve then, starting from the representation (5) for $\mathbf{r}$, establish expressions for the velocity vector $\mathbf{v}$, acceleration vector $\mathbf{a}$, and speed v of the particle.

(b) *(6 Points)* Assuming that $\dot{x} > 0$, verify that the following expression for $\mathbf{e}_t$ is correct:

$$\mathbf{e}_t = \frac{1}{\sqrt{2 - 2cx^2 + c^2x^4}}\left(\mathbf{E}_x + \left(1 - cx^2\right)\mathbf{E}_y\right), \qquad \mathbf{e}_n = \frac{\operatorname{sgn}(-2cx)}{\sqrt{2 - 2cx^2 + c^2x^4}}\left(\mathbf{E}_y - \left(1 - cx^2\right)\mathbf{E}_x\right). \tag{6}$$

Sketch the Serret-Frenet triad $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$ at the following three locations on the curve: $x = -1/\sqrt{c}$, $x = 0$, and $x = 1/\sqrt{c}$.

(c) *(6 Points)* Draw a free-body diagram of the particle assuming that it is in motion on the curve.

(d) *(7 Points)* Suppose that the particle is in motion on the curve. Show that the motion of the particle is governed by the differential equation

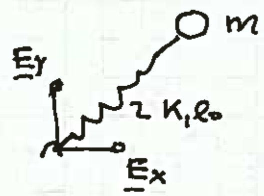
$$m\dot{v} = -\frac{mg\left(1 - cx^2\right)}{?} - \mu_k \|\mathbf{N}\| \frac{v}{|v|}. \tag{7}$$

For full credit supply the missing term. In addition, show that the normal force acting on the particle is

$$\mathbf{N} = \left(m\kappa v^2 + \frac{mg\operatorname{sgn}(-2cx)}{\sqrt{2 - 2cx^2 + c^2x^4}}\right)\mathbf{e}_n. \tag{8}$$

(e) *(5 Points)* Show that the particle will lift off the curve at $x = 1/\sqrt{c}$ if the velocity v of the particle at that point is greater than $\sqrt{\frac{g}{2c}}$.

$\boxed{\text{QUESTION 1}}$



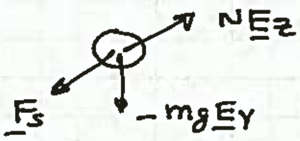
(a) $\underset{\sim}{r} = r\underset{\sim}{e}_r + f\,\underset{\sim}{E}_Y$

$\underset{\sim}{v} = \dot{r}\underset{\sim}{e}_r + r\dot{\theta}\underset{\sim}{e}_\theta + \dot{f}\,\underset{\sim}{E}_Y$

$\underset{\sim}{a} = \ddot{r}\underset{\sim}{e}_r + \dot{r}\dot{\theta}\underset{\sim}{e}_\theta + \dot{r}\dot{\theta}\underset{\sim}{e}_\theta + r\ddot{\theta}\underset{\sim}{e}_\theta - r\dot{\theta}^2\underset{\sim}{e}_r + \ddot{f}\,\underset{\sim}{E}_Y$

$= (\ddot{r} - r\dot{\theta}^2)\underset{\sim}{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underset{\sim}{e}_\theta + \ddot{f}\,\underset{\sim}{E}_Y$

(b)

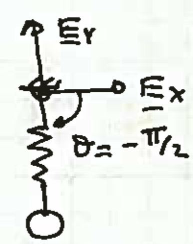


$\underset{\sim}{F}_s = -K(r - l_0)\underset{\sim}{e}_r$

(c) $\underset{\sim}{F} = m\underset{\sim}{a}$ $\qquad \underset{\sim}{F}_s + N\underset{\sim}{E}_z - mg\underset{\sim}{E}_Y = m\ddot{f}\underset{\sim}{E}_Y + m(\ddot{r} - r\dot{\theta}^2)\underset{\sim}{e}_r$
$+ m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\underset{\sim}{e}_\theta$

- $\underset{\sim}{e}_r$ $\quad m(\ddot{r} - r\dot{\theta}^2) = -m(g + \ddot{f})\sin\theta - K(r - l_0)$ $\qquad (\underset{\sim}{e}_r \cdot \underset{\sim}{E}_Y = \sin\theta)$
- $\underset{\sim}{e}_\theta$ $\quad m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -m(g + \ddot{f})\cos\theta$ $\qquad (\underset{\sim}{e}_\theta \cdot \underset{\sim}{E}_Y = \cos\theta)$

(d) If A has a constant velocity $\ddot{f} = 0$



Let $\dot{r}_0 = 0, \quad r = r_0, \quad \dot{\theta} = 0, \quad \theta = -\pi/2$

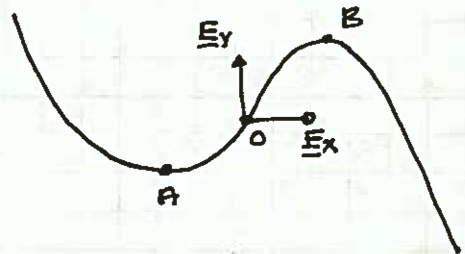
From eom $\quad \ddot{r} = +mg - K(r_0 - l_0) = 0$

$r_0\ddot{\theta} = 0$

$\Rightarrow$ $\ddot{r}$ and $\ddot{\theta}$ are zero so $\dot{r}$ and $\dot{\theta}$ remain 0, so r and θ remain constant

Hence particle will remain @ rest.

QUESTION 2



$$\underline{r} = x\,\underline{E}_X + \left(x - \frac{c}{3}x^3\right)\underline{E}_Y$$

$$A = \left(-\frac{1}{\sqrt{c}},\ \frac{1-3c}{3(\sqrt{c})^3}\right)$$

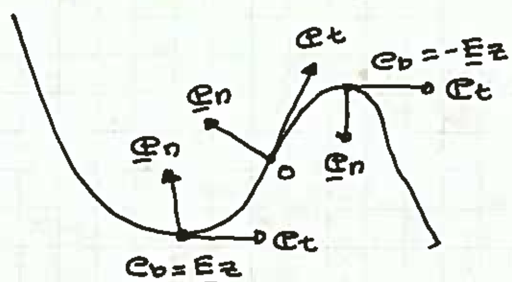
$$B = \left(\frac{1}{\sqrt{c}},\ \frac{3c-1}{3(\sqrt{c})^3}\right)$$

(a) $\underline{v} = \dot{\underline{r}} = \dot{x}\,\underline{E}_X + (\dot{x} - cx^2\dot{x})\underline{E}_Y$

$\underline{a} = \ddot{\underline{r}} = \ddot{x}\,\underline{E}_X + (\ddot{x} - cx^2\ddot{x})\underline{E}_Y - 2cx\dot{x}^2\,\underline{E}_Y$

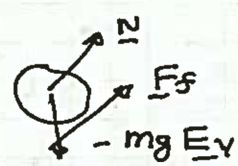
$v = \|\underline{v}\| = |\dot{x}|\sqrt{1 + (1-cx^2)^2} = |\dot{x}|\sqrt{2 + c^2x^4 - 2cx^2}$

(b) $\underline{e}_t = \dfrac{\underline{v}}{v} = \dfrac{\underline{E}_X + (1-cx^2)\underline{E}_Y}{\sqrt{2 + c^2x^4 - 2cx^2}}$



At O, $\kappa = 0$ and $\underline{e}_n$ is not uniquely defined we choose $\underline{e}_n$ such that $\underline{e}_b = \underline{E}_Z$ @ O.

(c)



$\underline{N} = N_n\,\underline{e}_n + N_b\,\underline{e}_b$

$\underline{F}_f = -\mu_K\|\underline{N}\|\,\underline{e}_t\left(\dfrac{v}{|v|}\right)$

(d) $\underline{F} = m\underline{a}$ $\qquad \underline{a} = \dot{v}\underline{e}_t + \kappa v^2 \underline{e}_n$

$$(\underline{F} = m\underline{a})\cdot \underline{e}_t = m\dot{v} = -mg\underline{E}_y\cdot\underline{e}_t - \mu_K \|\underline{N}\| \frac{v}{|v|}$$

$$= -mg\left(\frac{1-cx^2}{\sqrt{2-2cx^2+c^2x^4}}\right) - \mu_K\|\underline{N}\|\frac{v}{|v|}$$

$$(\underline{F}=m\underline{a})\cdot\underline{e}_n \Rightarrow m\kappa v^2 = -mg\,\underline{e}_n\cdot\underline{E}_y + N_n$$

$$(\underline{F}=m\underline{a})\cdot\underline{e}_b \Rightarrow 0 = -mg\underbrace{\underline{e}_b\cdot\underline{E}_y}_{=0} + N_b$$

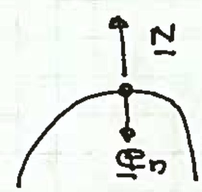
Hence $\underline{N} = N_n\underline{e}_n + N_b\underline{e}_b = N_n\underline{e}_n$

$$= (mg\,\underline{e}_n\cdot\underline{E}_y)\underline{e}_n + m\kappa v^2\underline{e}_n$$

$$= \left(m\kappa v^2 + \frac{mg\,\mathrm{sgn}(-2cx)}{\sqrt{2-2cx^2+c^2x^4}}\right)\underline{e}_n$$

(e) At B: $x = \frac{1}{\sqrt{c}}$, $\kappa = 2c$ $\qquad \mathrm{sgn}(-2cx) = -1$

$$\underline{N} = m\left(2cv^2 - \frac{mg}{1}\right)\underline{e}_n$$



For loss of contact @ B $\qquad N_n \ngtr 0$

Hence $\qquad v^2 > \frac{g}{2c}$