First Midterm Examination
Wednesday October 4 2000
Closed Books and Closed Notes
All Questions Carry Equal Points

Question 1
A Particle on a Circular Track

As shown in Figure 1, a bead of mass m moves on a rough circular track of radius R. The circular track is subject to a vertical motion

$$z = A\sin(\Omega t), \tag{1}$$

where A and Ω are constant. A vertical gravitational force acts on the particle.

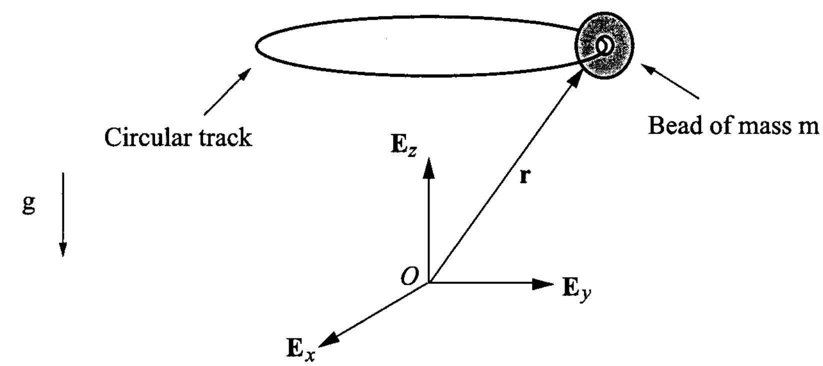


Figure 1: A bead of mass m moving on a circular track.

(a) Using a cylindrical polar coordinate system, show that the acceleration vector of the particle is

$$\mathbf{a} = -R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_\theta - A\Omega^2\sin(\Omega t)\mathbf{E}_z. \tag{2}$$

(b) What is the velocity vector $\mathbf{v}_{rel}$ of the particle relative to the circular track?

(c) Draw a free–body diagram of the particle.

(d) Show that the equations governing the motion of the particle relative to the circular track are

$$mR\ddot{\theta} = -\mu_d\|\mathbf{N}\|\frac{\dot{\theta}}{|\dot{\theta}|} \tag{3}$$

where

$$\mathbf{N} = -mR\dot{\theta}^2\mathbf{e}_r + (mg - mA\Omega^2\sin(\Omega t))\mathbf{E}_z. \tag{4}$$

(e) Suppose that the particle is not moving relative to the circular track. For this case, show that the force $\mathbf{P}$ exerted by the track on the particle is

$$\mathbf{P} = (mg - mA\Omega^2\sin(\Omega t))\mathbf{E}_z. \tag{5}$$

Give an explanation as to why the friction force is zero.

Question 2

Weightlessness

In the training of astronauts, a *KC–135* plane is flown in a parabolic arc which ranges from 32,000 feet to 36,000 feet in height over the surface of the Earth. For several minutes during this flight, astronauts experience "weightlessness."

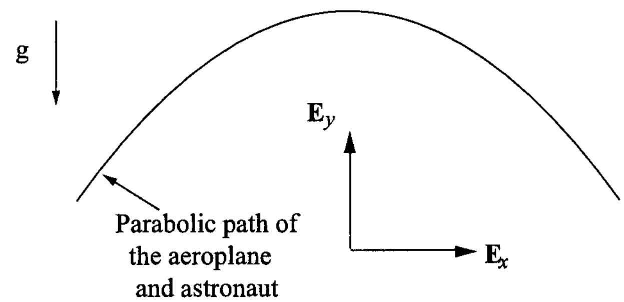


Figure 2: The parabolic path of the *KC–135* above the Earth's surface.

In this problem, the astronaut is modeled as a particle of mass m which is lying on the floor of the cabin of the aeroplane.

(a) Derive expressions for the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ vectors of the astronaut. Your starting point should be to assume the following expression for the position vector of the astronaut:

$$\mathbf{r} = x\mathbf{E}_x + (-\frac{\alpha}{2}x^2 + \beta)\mathbf{E}_y, \tag{6}$$

where α and β are positive constants.

(b) Assuming that $\dot{x}$ is positive, show that the unit tangent vector to the path of the astronaut and the speed of the astronaut are

$$\mathbf{e}_t = \frac{1}{\sqrt{1+\alpha^2x^2}}\left(\mathbf{E}_x - \alpha x\mathbf{E}_y\right), \quad v = \dot{x}\sqrt{1+\alpha^2x^2}. \tag{7}$$

(c) Draw a free–body diagram of the astronaut.

(d) Show that the force $\mathbf{R}$ exerted by the aeroplane on the astronaut is

$$\mathbf{R} = m(g - \alpha(\dot{x}^2 + x\ddot{x}))\mathbf{E}_y + m\ddot{x}\mathbf{E}_x. \tag{8}$$

(e) The normal vector $\mathbf{e}_n$ to the astronaut's path and the plane's path is

$$\mathbf{e}_n = -\frac{1}{\sqrt{1+\alpha^2x^2}}\left(\mathbf{E}_y + \alpha x\mathbf{E}_x\right). \tag{9}$$

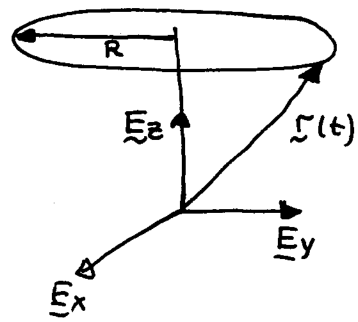
Using this result, show that the astronaut will start to lose contact with the floor of the aeroplane when

$$\dot{x}^2 = \frac{g}{\alpha}. \tag{10}$$

You may assume that the normal to the floor of the aeroplane where the astronaut is located is parallel to $\mathbf{e}_n$.

Question 1

(All 5 Parts Each worth 5 points)



(a)

$$\mathbf{r} = R\mathbf{e}_r + z\mathbf{E}_z$$

$$= R\mathbf{e}_r + A\sin\Omega t\,\mathbf{E}_z$$

$$\mathbf{v} = R\dot{\mathbf{e}}_r + A\Omega\cos\Omega t\,\mathbf{E}_z$$

$$= R\dot{\theta}\mathbf{e}_\theta + A\Omega\cos\Omega t\,\mathbf{E}_z \qquad (\text{using } \dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta)$$

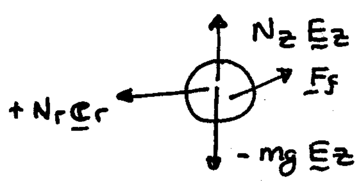
$$\mathbf{a} = R\ddot{\theta}\mathbf{e}_\theta + R\dot{\theta}\dot{\mathbf{e}}_\theta + A\Omega^2(-\sin\Omega t)\mathbf{E}_z$$

$$= R\ddot{\theta}\mathbf{e}_\theta - R\dot{\theta}^2\mathbf{e}_r + A\Omega^2(-\sin\Omega t)\mathbf{E}_z \qquad (\text{using } \dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r)$$

(b)

$$\mathbf{v}_{rel} = R\dot{\theta}\mathbf{e}_\theta \qquad (= \mathbf{v} - \mathbf{v}_{track}\text{, where } \mathbf{v}_{track} = A\Omega\cos\Omega t\,\mathbf{E}_z)$$

(c)



$$\mathbf{F}_f = -\mu_d\|\mathbf{N}\|\frac{\mathbf{v}_{rel}}{\|\mathbf{v}_{rel}\|} \text{ or } F_{f\theta}\mathbf{e}_\theta$$

$$\mathbf{N} = N_r\mathbf{e}_r + N_z\mathbf{E}_z$$

(d) $\mathbf{F} = m\mathbf{a}$:

$$(-mg + N_z)\mathbf{E}_z - \mathbf{F}_f + N_r\mathbf{e}_r = m\mathbf{a}$$

$\cdot\,\mathbf{e}_\theta$:
$$mR\ddot{\theta} = -\mu_d\|\mathbf{N}\|\frac{\mathbf{v}_{rel}\cdot\mathbf{e}_\theta}{\|\mathbf{v}_{rel}\|} = -\mu_d\|\mathbf{N}\|\frac{R\dot{\theta}}{|R\dot{\theta}|}$$

$\cdot\,\mathbf{e}_r$:
$$-mR\dot{\theta}^2 = N_r$$

$\cdot\,\mathbf{E}_z$:
$$-mA\Omega^2\sin\Omega t = N_z - mg$$

$$\Rightarrow \mathbf{N} = -mR\dot{\theta}^2\mathbf{e}_r + (mg - mA\Omega^2\sin\Omega t)\mathbf{E}_z$$

(e) In this case $\mathbf{v}_{rel} = \mathbf{0} \Rightarrow \dot{\theta} = 0$ and $\ddot{\theta} = 0. \Rightarrow \mathbf{a} = -A\Omega^2\sin\Omega t\,\mathbf{E}_z$

In addition, $\mathbf{F}_f = F_{f\theta}\mathbf{e}_\theta$ where $F_{f\theta}$ is unknown (i.e., we have static friction)

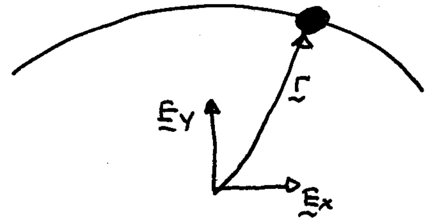
The force exerted by the track on the particle $\mathbf{P} = \mathbf{F}_f + \mathbf{N} = m\mathbf{a} + mg\mathbf{E}_z$

$$= m(g - A\Omega^2\sin\Omega t)\mathbf{E}_z.$$

The friction force is zero because there is no external applied force in the $\mathbf{e}_\theta$ direction and no acceleration in the $\mathbf{e}_\theta$ direction.

Question 2

(All 5 Parts each worth 5 Points).



(a) $$\underset{\sim}{r} = x\underset{\sim}{E}_x + \left(-\frac{\alpha}{2}x^2 + B\right)\underset{\sim}{E}_y$$

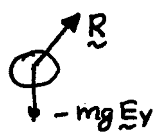
$$\Rightarrow \quad \underset{\sim}{v} = \dot{x}\left(\underset{\sim}{E}_x - \alpha x \underset{\sim}{E}_y\right) = \dot{\underset{\sim}{r}}$$

$$\Rightarrow \quad \underset{\sim}{a} = \ddot{x}\left(\underset{\sim}{E}_x - \alpha x \underset{\sim}{E}_y\right) - \alpha \dot{x}^2 \underset{\sim}{E}_y = \dot{\underset{\sim}{v}}$$

(b) $$\underset{\sim}{v} = v\underset{\sim}{e}_t \quad \Rightarrow \quad v = \|\underset{\sim}{v}\| = \sqrt{\dot{x}^2 + \alpha^2 x^2 \dot{x}^2} = \dot{x}\sqrt{1+\alpha^2 x^2}$$

$$\Rightarrow \quad \underset{\sim}{e}_t = \frac{\underset{\sim}{v}}{\|\underset{\sim}{v}\|} = \frac{\underset{\sim}{E}_x - \alpha x \underset{\sim}{E}_y}{\sqrt{1+\alpha^2 x^2}}$$

(c)



$$\underset{\sim}{R} = R_x\underset{\sim}{E}_x + R_y\underset{\sim}{E}_y + R_z\underset{\sim}{E}_z$$
$$= R_n\underset{\sim}{e}_n + R_t\underset{\sim}{e}_t + R_b\underset{\sim}{e}_b$$

(d) $$\underset{\sim}{F} = m\underset{\sim}{a} \quad \Rightarrow \quad \underset{\sim}{R} = mg\underset{\sim}{E}_y + m\underset{\sim}{a}$$

$$= \left(mg - \alpha m(\dot{x}^2 + x\ddot{x})\right)\underset{\sim}{E}_y + m\ddot{x}\underset{\sim}{E}_x$$

(e) To loose contact $\underset{\sim}{R}\cdot\underset{\sim}{e}_n = 0$

Now, $$\underset{\sim}{R}\cdot\underset{\sim}{e}_n = -\left(\sqrt{1+\alpha^2x^2}\right)^{-1}\left(mg - \alpha m(\dot{x}^2 + x\ddot{x}) + \alpha x \ddot{x} m\right)$$
$$= -\left(\sqrt{1+\alpha^2x^2}\right)^{-1}\left(mg - \alpha m \dot{x}^2\right)$$

$$\Rightarrow \quad \underset{\sim}{R}\cdot\underset{\sim}{e}_n = 0 \quad \Rightarrow \quad mg = \alpha m \dot{x}^2$$

$$\Rightarrow \quad \dot{x}^2 = \frac{g}{\alpha}$$

Alternatively if one assumes $\underset{\sim}{R} = \underset{\sim}{0}$, then, in addition to $\underset{\sim}{R}\cdot\underset{\sim}{e}_n = 0$, we have $\underset{\sim}{R}\cdot\underset{\sim}{e}_t = 0$. Now $\underset{\sim}{R}\cdot\underset{\sim}{e}_t = 0$ will yield $\ddot{x} = 0$. as an extra condition.